INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 22, 325–352 (1996)

ARTIFICIAL BOUNDARIES AND FLUX AND PRESSURE CONDITIONS FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

JOHN G. HEYWOOD

Mathematics Department, University of British Columbia, 121–1984 Mathematics Road, Vancouver, B.C. V6T 1Y4, Canada

AND

ROLF RANNACHER AND STEFAN TUREK

Institut für Angewandte Mathematik, Universität Heidelberg, In Neuenheimer Feld 293, D-69120 Heidelberg, Germany

SUMMARY

Fluid dynamical problems are often conceptualized in unbounded domains. However, most methods of numerical simulation then require a truncation of the conceptual domain to a bounded one, thereby introducing artificial boundaries. Here we analyse our experience in choosing artificial boundary conditions implicitly through the choice of variational formulations. We deal particularly with a class of problems that involve the prescription of pressure drops and/or net flux conditions.

KEY WORDS: Navier–Stokes equations; artificial boundary conditions; flux and pressure conditions; finite elements

1. INTRODUCTION

Most flow problems of scientific or engineering interest, such as flows past obstacles, around corners or through pipes or apertures, are first conceptualized in unbounded domains. This is an idealization intended to focus on a phenomenon of interest, free of the effects of distant boundaries. We begin this paper by reviewing the mathematical formulations for unbounded domains of a class of problems that involve the prescription of pressure drops and/or net flux conditions. These formulations are suggestive of analogous formulations for bounded domains, which are appropriate when a bounded domain is obtained as the truncation of an unbounded domain for the purpose of making a numerical computation.

We focus particularly on variational formulations rather than on their classical counterparts. The principal issue concerning variational formulations is not a choice of boundary conditions but a choice of function spaces. This choice of function spaces, however, seems relatively straightforward in comparison with choosing boundary conditions. We accept what seems to be the simplest and most natural choice for these function spaces, namely that which leaves functions as free as possible, and investigate the consequences through numerical experiments and by drawing out the relationship between the resulting variational problems and the 'artificial' boundary conditions that are implicit in them.

We often refer to the boundary conditions arrived at in this way as 'do nothing' boundary conditions, since we do not try to achieve any special effect or boundary condition through restrictions

CCC 0271–2091/96/050325–28

Received 7 July 1992
Revised 28 December 1994

in the function spaces. As it turns out, these boundary conditions are the same as those that have already been recommended by Gresho (see e.g. Reference 1) for use along outflow boundaries (for more recent references see Reference 2). In our variational formulations these boundary conditions are implicitly combined with 'net flux' and/or 'pressure drop' conditions and applied equally along both inflow and outflow boundaries. This allows us to consider two types of problems that we show to be dual to each other: find certain net fluxes (say through individual pipes in a network of pipes) from prescribed pressure drops; alternatively, find the pressure drops that produce these net fluxes.

There are currently many possible choices of outflow boundary conditions under consideration by the computational community, without any completely clear criteria for preferring one over another. Gresho's contention that these 'do nothing' boundary conditions are probably the best possible general-purpose boundary conditions for use along outflow boundaries seems to be supported from the mathematical point of view by their simplicity and elegance within the variational framework.

Interestingly enough, these boundary conditions have not received much attention from the mathematical community. However, as it now seems clear that they have great practical importance, it is apparent that they deserve serious mathematical investigation as part of the general Navier–Stokes theory. To that end we close this paper by offering what we can in the way of theorems of existence, uniqueness, continuous dependence and stability and draw attention to several points of difficulty that limit our theorems in comparison with what is known in the case of Dirichlet boundary conditions.

Our interest in these matters was stimulated by our experience in testing a two-dimensional finite element code of Turek[3] which is based on the use of discretely divergence-free finite elements. Because it uses divergence-free elements, it is natural to formulate problems for this code in the same way as they are usually formulated by mathematicians, namely as pressure-free variational problems for the velocity using a test space of divergence-free functions. Our analysis begins with formulations of this type. However, we also derive equivalent formulations in terms of both the velocity and pressure as primary variables and also for two-dimensional situations in terms of the streamfunction. The 'do nothing' approach leads to the same result in each case. Of course, the mathematical questions studied in this paper are relevant to all methods of simulating viscous incompressible flow subject to these boundary conditions, whatever the means of enforcing them.

The contents of this paper are as follows. As already mentioned, in Section 2 we review the theory of properly posing problems that involve flux and pressure conditions in unbounded domains. In Section 3 we give analogous variational formulations for the case of bounded domains and report on our computational experience with them. In section 4 we draw out the relationship between these variational formulations and the corresponding classical formulations in terms of the velocity and pressure. In Section 5 we give equivalent variational formulations in terms of the streamfunction. That, of course, is restricted to the two-dimensional case. In Section 6 we present the beginnings of a mathematical theory for these problems in both the two- and three-dimensional cases and for both the stationary and non-stationary equations.

2. NAVIER–STOKES PROBLEMS IN UNBOUNDED DOMAINS

As mentioned above, many problems in fluid dynamics are conceptualized and studied mathematically in unbounded domains. For example, in studying flow past an obstacle, one would usually like to determine the asymptotic structure of the wake and the force on the obstacle free of the influence of distant boundaries. Another example, the one that we are particularly interested in here, concerns fluid jets. To fix ideas, consider a plane wall which has a hole in it and let the flow region be its complement. We call this an aperature domain; see Figure 1. It is a natural problem in an aperture domain to study a jet of fluid that is driven through the aperture by a drop in the pressure from one side of the wall to the

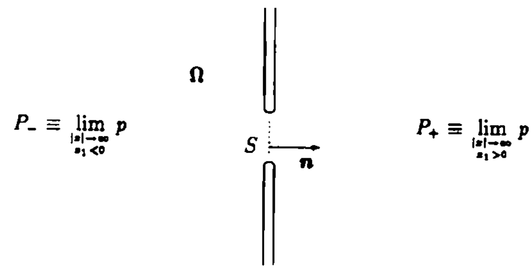

Figure 1. Notation for flow through an aperture in an infinite wall

other. To make the drop in pressure quantitatively precise, one may prove first that the pressure must tend to a limit at infinity in each half-space and then consider the difference in these limits.

Thus the problem of finding a jet through an aperature may be formulated in classical terms by supplementing the usual initial-boundary value problem for the Navier–Stokes equations (with Dirichlet boundary conditions),

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1a}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{u}(x,t) \to 0 \quad \text{as } |x| \to \infty, \tag{1b}$$

with an auxiliary condition on the pressure,

$$\lim_{\substack{|x|\to\infty \\ x_1<0}} p(x,t) - \lim_{\substack{|x|\to\infty \\ x_1>0}} p(x,t) = P(t), \tag{2}$$

where $P(t)$ is prescribed.

As it happens, there is another equally good way of determining such jets. Instead of prescribing the drop in pressure, one may prescribe the net flux through the aperture. That is, one can replace the auxiliary pressure condition (2) by the auxiliary flux condition

$$\int_S \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = F(t), \tag{3}$$

where $F(t)$ is prescribed.

Let us turn now to the variatonal formulations of these problems. Consider first the initial boundary value problem (1) without the auxiliary conditions (2) or (3). In the older mathematical literature it is often posed for arbitrary domains Ω, bounded or unbounded, as follows: Find $\boldsymbol{u}(t)$ satisfying the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ such that for all $t > 0$

$$\boldsymbol{u}(t) \in \boldsymbol{J}_1^*(\Omega) \equiv \{\varphi \in \mathbf{W}_2^1(\Omega) : \varphi|_{\partial\Omega} = 0,\ \nabla\cdot\varphi = 0\}, \tag{4a}$$

$$\nu(\nabla\boldsymbol{u}, \nabla\varphi) + (\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) = 0, \quad \forall\varphi \in \boldsymbol{J}_1^*(\Omega). \tag{4b}$$

Here $(\cdot,\cdot)$ denotes the inner product in $\boldsymbol{L}^2(\Omega)$ and $\mathbf{W}_2^1(\Omega)$ denotes the Sobolev space consisting of functions that belong to $\boldsymbol{L}^2(\Omega)$ and have first-order spatial derivatives in $\boldsymbol{L}^2(\Omega)$. We are using bold face to indicate $\mathbb{R}^n$-valued functions and function spaces.

Elsewhere in the older literature the same problem (1) is formulated slightly differently as follows: find $\boldsymbol{u}(t)$ satisfying the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ such that for all $t > 0$

$$\boldsymbol{u}(t) \in \boldsymbol{J}_1(\Omega) \equiv \text{completion of } \boldsymbol{D}(\Omega) \text{ in } \mathbf{W}_1^2(\Omega), \tag{5a}$$

$$\nu(\nabla\boldsymbol{u}, \nabla\varphi) + (\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) = 0, \quad \forall\varphi \in \boldsymbol{J}_1(\Omega). \tag{5b}$$

Here $D(\Omega) \equiv \{\varphi \in C_0^\infty(\Omega): \nabla\cdot\varphi = 0\}$, where $C_0^\infty(\Omega)$ is the set of all smooth functions with compact supports in Ω (i.e. smooth functions vanishing near the boundary and near infinity). The completion of $D(\Omega)$ in $W_1^2(\Omega)$ consists of those elements of $W_1^2(\Omega)$ which can be approximated arbitrarily closely by elements of $D(\Omega)$ in the norm

$$\|u\|_{W_1^2(\Omega)} = \left(\int_\Omega (|u|^2+|\nabla u|^2)\,dx\right)^{1/2}.$$

It was originally thought that the spaces $J_1^*(\Omega)$ and $J_1(\Omega)$ are the same and that consequently these two formulations of problem (1) are the same. In fact, for certain classes of domains, the three-dimensional aperture domain being a prototypical example, these function spaces are different and neither of the formulations (4) or (5) correctly represents problem (1). Instead, hidden within the formulation (4) is an auxiliary condition on the pressure, namely that the pressure drop must be zero in the sense of condition (2), and hidden within the formulation (5) is another different auxiliary condition, namely that the net flux through the aperture must be zero in the sense of condition (3).

Indeed, if we take an element φ of $D(\Omega)$ and apply the divergence theorem to it in the left half-space, we see that

$$\int_S \varphi\cdot n\,dS = \int_{x_1<0} \nabla\cdot\varphi\,dx = 0.$$

It follows that elements of $J_1(\Omega)$, being limits of functions in $D(\Omega)$, must also have zero net flux through the aperture and thus this condition is also contained in the formulation (5).

The analysis of the formulation (4) is more involved. Let us consider only the case of a three-dimensional aperture domain. Then one may construct an explicit function b in $J_1^*(\Omega)$ that carries a non-trivial net flux through the aperture and normalize it by requiring that

$$\int_S b\cdot n\,dS = 1.$$

This of course establishes that the two spaces $J_1(\Omega)$ and $J_1^*(\Omega)$ are different. Further, it can be proven that the only real difference between these two function spaces is the single flux carrier b. More precisely, $J_1(\Omega)$ is contained in $J_1^*(\Omega)$, while on the other hand every element φ of $J_1^*(\Omega)$ can be written as $\varphi = Fb + \psi$, where ψ is some element of $J_1(\Omega)$ and $F = \int_S \varphi\cdot n\,dS$.

The original intention for setting the condition (4b) in posing the Dirichlet problem was to insure that there is a scalar function p such that $-\nabla p = u_t + u\cdot\nabla u - \nu\Delta u$. However, for that it is enough to test with test functions φ belonging to $J_1(\Omega)$ or even to its dense subset $D(\Omega)$. When we test with all φ in $J_1^*(\Omega)$, that includes a test with the flux carrier b. This extra test is in fact a test of the pressure drop. It can be shown that (4b) holds with $\varphi = b$ if and only if the pressure drop is zero. Thus the variational formulation (4) of problem (1) actually contains the 'hidden' condition that the pressure drop from one side of the wall to the other must be zero.

It remains now to generalize the variational formulations (4) and (5) so as to intentionally incorporate prescribed values of the pressure drop $P(t)$ in (2) or of the net flux $F(t)$ in (3). We will refer to the literature for the rigorous analysis and simply state here the final results.

The correct variational formulation of the prescribed pressure drop problem (1), (2) is: Find $u(t)$ satisfying the initial condition $u|_{t=0} = u_0$ such that for all $t > 0$

$$u(t) \in J_1^*(\Omega), \tag{6a}$$

$$\nu(\nabla u, \nabla\varphi) + (u_t + u\cdot\nabla u, \varphi) = -P(t)\int_S \varphi\cdot n\,dS, \quad \forall\varphi \in J_1^*(\Omega). \tag{6b}$$

The correct variational formulation of the prescribed net flux problem (1), (3) is: Find $\boldsymbol{u}(t) = F(t)\boldsymbol{b} + \boldsymbol{v}(t)$ satisfying the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$ such that for all $t > 0$

$$\boldsymbol{v}(t) \in \boldsymbol{J}_1(\Omega), \tag{7a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{J}_1(\Omega). \tag{7b}$$

Perhaps it should be pointed out that, having constructed $\mathbf{b}$, the real unknown in (7b) is $\mathbf{v}$ and that equation (7b) can be equivalently written as

$$\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \mathbf{b} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \mathbf{b}, \boldsymbol{\varphi}) = -\nu(\nabla \mathbf{b}, \nabla \boldsymbol{\varphi}) - (\mathbf{b}_t + \mathbf{b} \cdot \nabla \mathbf{b}, \boldsymbol{\varphi}).$$

The results that we have described in this section are from the work of Heywood,[4,5] which initiated a general study of the relationship between the geometry of unbounded domains and the auxiliary conditions that are needed to formulate well-posed problems for the Navier–Stokes equations. Further results and references can be found in the works of Solonnikov,[6] Maslennikova and Bogovskii[7] and Galdi.[8,9] These investigations all depend in an essential way on an analysis of the function spaces that enter into the variational formulations of these problems. One notable result, already given in Reference 4, is that $\boldsymbol{J}_1^*(\Omega) = \boldsymbol{J}_1(\Omega)$ in the case of an exterior domain. Consequently, pressure drops cannot be prescribed in an exterior domain and solutions of the initial value (Dirichlet) problem (1) are uniquely determined without them. In particular, flow past an obstale in an exterior domain must be driven by the prescription of a non-zero limit for the velocity at infinity. Thus there are fundamentally different mechanisms that drive non-trivial flows in different types of unbounded domains. This paper concerns the truncation to bounded domains of flows which, in the idealization of an unbounded domain, are driven by pressure drops.

What we are going to do now is change the point of view to that of the computational practitioner and use the theory from this section as guidance in formulating problems.

3. FLUX AND PRESSURE CONDITIONS IN BOUNDED DOMAINS

To fix ideas in a familiar setting with which we can make later comparisons, let us begin by considering a common test problem, that of calculating non-steady flow past an obstable (here taken as an inclined ellipse) situated in a rectangle; see Figure 2.

The velocity $\boldsymbol{u}(t)$ is required to be zero on the upper and lower boundaries and on the surface of the ellipse, while a parabolic 'Poiseuille' inflow profile is prescribed on the upstream boundary. We denote by Γ the union of those portions of the boundary on which Dirichlet conditions are imposed. Rather than giving serious thought to the downstream boundary condition on S, in seeking a variational formulation, one can simply decide to 'do nothing', i.e. leave the solution and the test space free on that portion of the boundary.

To give a variational formulation of this problem using *solenoidal spaces* (meaning spaces consisting of solenoidal functions), the first step is to construct a solenoidal extension $\mathbf{b}$ of the prescribed Dirichlet boundary values into the whole of the domain Ω. Note that since $\mathbf{b}$ is to be solenoidal, it must carry the incoming flux at the left boundary through the domain and out across the

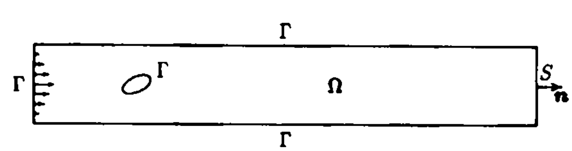

Figure 2. Notation for a flow region having an artificial boundary at the outlet S

downstream boundary. The construction of such a function **b** might appear to be a difficult task. For aid in the construction of a flux carrier in the continuous case, the reader may consult Theorem 3.1 and Exercise 3.4 in Chap. III of Reference 8. Fortunately, in computations involving divergence-free finite elements, the construction of **b** can be achieved by simply prescribing the appropriate nodal values along the boundary Γ. This procedure automatically generates a discretely divergence-free extension **b** of the boundary values having support in a one-element-wide strip along the boundary. While in practice one need not be conscious of this, we need to realize that it is being done in order to analyse the method and the variational formulations behind it.

Having constructed a solenoidal extension **b** of the boundary values, a variational formulation of the problem indicated by Figure 2, using solenoidal vector fields, is obtained by requiring $\boldsymbol{u}(t) = \mathbf{b} + \boldsymbol{v}(t)$, where for all t

$$\boldsymbol{v}(t) \in \boldsymbol{J}_1^*(\Omega) \equiv \{\boldsymbol{\varphi} \in \mathbf{W}_2^1(\Omega): \boldsymbol{\varphi}|_\Gamma = 0, \nabla \cdot \boldsymbol{\varphi} = 0\}, \tag{8a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{J}_1^*(\Omega). \tag{8b}$$

Here, as in Section 2, we use bold face to indicate $\mathbb{R}^n$-valued functions and function spaces, $(.\,,.)$ denotes the inner product in $\boldsymbol{L}^2(\Omega)$ and $\mathbf{W}_2^1(\Omega)$ denotes the Sobolev space consisting of functions that belong to $\boldsymbol{L}^2(\Omega)$ and have first-order spatial derivatives in $\boldsymbol{L}^2(\Omega)$. In order to discuss both steady and non-steady problems simultaneously, we have omitted the initial condition in writing (8). Thus (8) represents the Navier–Stokes equations along with boundary conditions. The initial–boundary value problem is formulated by adding the initial condition $\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0$. The stationary problem is formulated by adding the condition that $\boldsymbol{u}_t = 0$. What we are mainly interested in is how the equations are combined with boundary conditions and other 'hidden' auxiliary conditions in variational formulations.

Corresponding to (8), which is a pressure-free formulation using solenoidal spaces, there is also an equivalent *standard* formulation which is expressed without reference to solenoidal spaces. For this formulation the extension **b** need not be solenoidal (again it can be constructed by simply assigning nodal values along the boundary). The requirement is then that $\boldsymbol{u}(t) = \mathbf{b} + \boldsymbol{v}(t)$, where for all t

$$\boldsymbol{v}(t) \in \boldsymbol{V}_1^*(\Omega) \equiv \{\boldsymbol{\varphi} \in \boldsymbol{W}_2^1(\Omega): \boldsymbol{\varphi}|_\Gamma = 0\}, \qquad p(t) \in L^2(\Omega), \tag{9a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}_1^*(\Omega), \tag{9b}$$

$$(\chi, \nabla \cdot \boldsymbol{u}) = 0, \quad \forall \chi \in L^2(\Omega). \tag{9c}$$

The results of our computations based on (8), like those reported by others on the basis of (9), show a truly remarkable 'transparency' of the downstream boundary when it is handled in this way (Figure 3). Testing by doubling the length of the computational domain is seen to make almost no discernible difference in the flow in the shorter common region. Figures 3(a)–3(c) are different representations of exactly the same computations.

As these results appear highly satisfactory, there seems little reason to ask about the boundary conditions that must be implicit in these variational formulations. However, now, to motivate such questions, let us consider low-Reynods-number flow through a junction in a system of pipes, again prescribing a Poiseuille inflow upstream. Figure 4 shows steady streamlines for computations based on the same variational formulations as above, each with the same inflow but with varying lengths of pipe beyond the junction.

There seems to be something of a puzzle here in that the flow through the junction is seen to be highly dependent on the positions of the artificial boundaries even if they are far from the junction. One

(a)

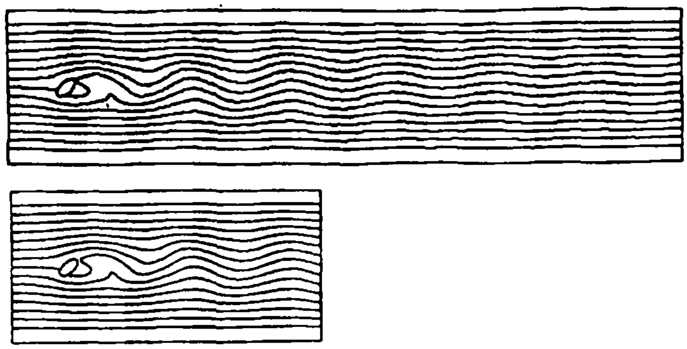

Figure 3a. Streamlines at $Re = 500$ after 100 time steps, starting from Stokes flow, with constant Poiseuille inflow, computed in domains of different lengths on the basis of the variational formulation (8). The flow is nearly identical in the shorter common region, indicating a satisfactory treatment of the downstream artificial boundary

(b)

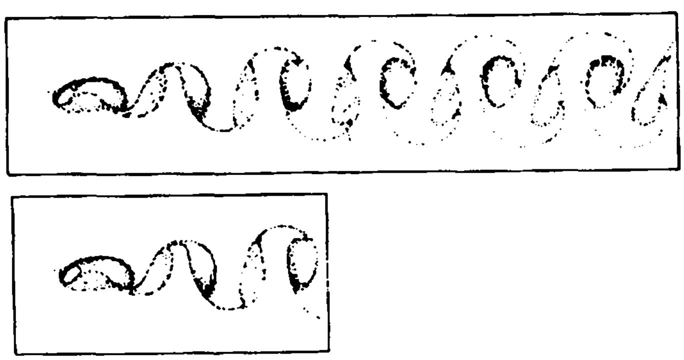

Figure 3(b). The same computations represented by particle tracing, showing von Kármán streets as usually visualised in physical experiments by smoke or the like

(c)

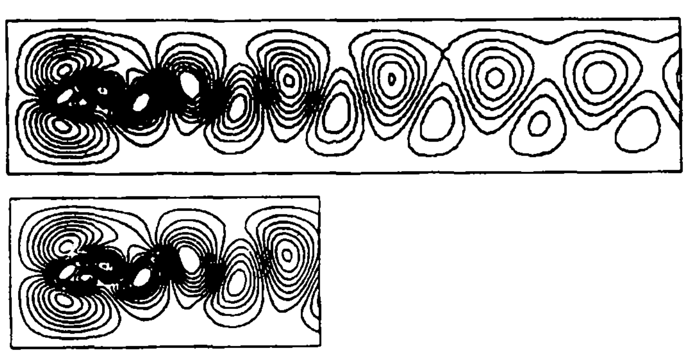

Figure 3(c). Relative streamlines for the same computations, showing the difference $u - \bar{u}$ between the non-linear solution u and the solution $\bar{u}$ of the *stationary* Stokes problem on the same domain Ω

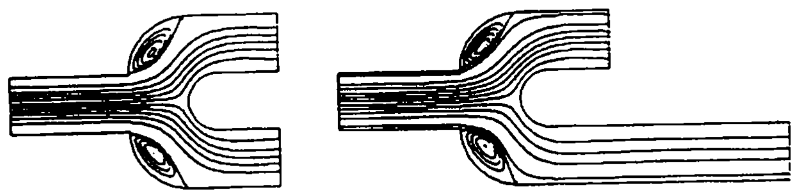

Figure 4. Streamlines of flow at $Re = 50$ with constant Poiseuille inflow, computed using the same variational formulation (8) as in Figure 3. The net flux through each outlet is highly dependent upon the relative lengths of the downstream sections

might wonder whether the variational formulation (8) or (9) has some 'hidden' condition within it analogous to the 'hidden' pressure condition (2) in the variational formulation (4) of problem (1). That is the point of this discussion. It does have a precisely analogous 'hidden' pressure condition. This can be seen be examining the 'natural' boundary conditions that are associated with the variational formulations (8) and (9), as we shall show in the next section. In particular, it will be seen that they imply that the mean pressure on each free section S_i (see Figure 5) is zero:

$$\frac{1}{|S_i|}\int_{S_i} p\,\mathrm{d}s = 0 \quad \text{for } i = 2, 3.$$

Thus in Figure 4 the pressure gradient is greater in the shorter of the two outflow sections, which explains why there is a greater flow through that section. This example suggests that we might consider formulating problems more generally in terms of *prescribed pressure drops* and that we need not distinguish between sections of inflow and outflow or even know which are which. For a flow region with multiple inlet/outlets as indicated in Figure 5, it seems natural to seek solutions for which the mean pressure over each outlet section is prescribed. Therefore let us consider the following.

Prescribed pressure drop problem. For any prescribed $P_i(t)$ find $\boldsymbol{u}(t)$ and $p(t)$ such that

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{10a}$$

$$\boldsymbol{u}|_\Gamma = 0, \qquad \frac{1}{|S_i|}\int_{S_i} p\,\mathrm{d}s = P_i(t). \tag{10b}$$

It is this type of problem that needs to be considered in order to determine the net flux through each of various inlets or outlets given the pressure drops between them. One can even ask whether there will be a positive net inflow or outflow through some particular duct for given prescribed values of the mean pressures. Notice that at this point we do not want to commit ourselves to any particular boundary conditions along the outlets S_i. Our only stated objective is to achieve prescribed differences between the mean pressures across the various outlets. It is implicit that we want to achieve this by whatever boundary conditions work best in some vague sense. We hope to find these by posing the

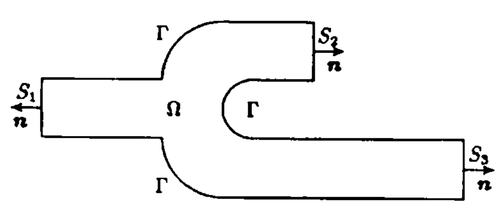

Figure 5. Notation for flow regions having artificial boundaries at multiple inlet/outlets S_i

problem variationally in the most natural possible way. What would that formulation be? For guidance we look to the analogous problem (1), (2) for unbounded domains. Its variational formulation (6) can be copied word for word.

Variational pressure drop problem (*with solenoidal spaces*). Find $\boldsymbol{u}(t)$ such that for all t

$$\boldsymbol{u}(t) \in \boldsymbol{J}_1^*(\Omega) \equiv \{\boldsymbol{\varphi} \in \boldsymbol{W}_2^1(\Omega): \boldsymbol{\varphi}|_\Gamma = 0, \nabla \cdot \boldsymbol{\varphi} = 0\}, \tag{11a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = -\sum_i P_i(t) \int_{S_i} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, \mathrm{d}S, \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{J}_1^*(\Omega). \tag{11b}$$

It is easy to see that a variational formulation such as this is mathematically well posed. It is somewhat more difficult to translate it precisely in terms of boundary conditions and the like. When one does, as in the next section, it will be seen that the conditions (11) imply something more along the free boundary S than was asked for in (10b). Therefore problem (10) by itself does not quite form a well-posed problem. We note too that a more general class of functionals can be introduced on the right side of (11b). Such more general problems are briefly considered at the end of Section 4. However, the simple case considered here seems to have a very wide range of useful applications. It is interesting that this problem, in which conditions for the pressure are prescribed, is so easily set in a pressure-free variational formulation. The analogue of problem (11) in terms of both primary variables is posed as follows.

Variational pressure drop problem (*without solenoidal spaces*). Find $\boldsymbol{u}(t)$ such that for all t

$$\boldsymbol{u}(t) \in \boldsymbol{V}_1^*(\Omega) \equiv \{\boldsymbol{\varphi} \in \mathbf{W}_2^1(\Omega): \boldsymbol{\varphi}|_\Gamma = 0\}, \qquad p(t) \in L^2(\Omega), \tag{12a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) = -\sum_j P_j(t) \int_{S_i} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, \mathrm{d}S, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{V}_1^*(\Omega), \tag{12b}$$

$$(\chi, \nabla \cdot \boldsymbol{u}) = 0, \quad \forall \chi \in L^2(\Omega). \tag{12c}$$

The prescription of pressure drops is not the only natural way of posing problems for flow through a system of ducts like that of Figure 5. Indeed one may wish to *find* the pressure drops that are required to achieve a desired *net flux* through each of various ducts. Thus we also consider the following.

Prescribed net flux problem. For any prescribed $F_i(t)$ satisfying $\sum_i F_i(t) = 0$ find $\boldsymbol{u}(t)$ and $p(t)$ such that

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{13a}$$

$$\boldsymbol{u}|_\Gamma = 0, \qquad \int_{S_i} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = F_i(t). \tag{13b}$$

We remark that as in posing problem (10) above we do not want to commit ourselves at this point to the details of the boundary conditions on S_i.

To incorporate these flux conditions into a variational formulation of the Navier–Stokes equations, we look again to the analogous problem (1), (3) for unbounded domains. Its variational formulation (7) can be copied exactly. One first constructs solenoidal flux carriers $\mathbf{b}_i$, $i \geqslant 2$, carrying a unit net flux from an arbitrarily chosen reference inlet/outlet S_1 to each of the others. Thus, if there are three inlet/outlets (see Figure 5), let $\mathbf{b}_i$, $i = 2, 3$, satisfy

$$\mathbf{b}_i \in \boldsymbol{J}_1^*(\Omega), \qquad \int_{S_1} \mathbf{b}_i \cdot \boldsymbol{n} \, \mathrm{d}S = -1, \qquad \int_{S_j} \mathbf{b}_i \cdot \boldsymbol{n} \, \mathrm{d}S = \delta_{ij}, \quad j = 2, 3. \tag{14}$$

Then an appropriate formulation is the following.

Variational net flux problem (with solenoidal spaces). Find $\boldsymbol{u}(t) = F_2(t)\mathbf{b}_2 + F_3(t)\mathbf{b}_3 + \boldsymbol{v}(t)$ such that for all t

$$\boldsymbol{v}(t) \in J_1(\Omega) \equiv \left\{ \varphi \in \mathbf{W}_2^1(\Omega) \colon \varphi|_\Gamma = 0, \nabla \cdot \varphi = 0, \int_{S_i} \varphi \cdot \boldsymbol{n} \, \mathrm{d}S = 0, \forall i \right\}, \tag{15a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \varphi) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \varphi) = 0, \quad \forall \varphi \in J_1(\Omega). \tag{15b}$$

If one is not using solenoidal spaces, the functions $\mathbf{b}_i$ are not required to be solenoidal and the appropriate formulation is as follows.

Variational net flux problem (without solenoidal spaces). Find $\boldsymbol{u}(t) = F_2(t)\mathbf{b}_2 + F_3(t)\mathbf{b}_3 + \boldsymbol{v}(t)$ and $p(t)$ such that for all t

$$\boldsymbol{v}(t) \in V_1(\Omega) \equiv \left\{ \varphi \in \mathbf{W}_2^1(\Omega) \colon \varphi|_\Gamma = 0, \int_{S_i} \varphi \cdot \boldsymbol{n} \, \mathrm{d}s = 0, \forall i \right\}, \qquad p(t) \in L^2(\Omega), \tag{16a}$$

$$\nu(\nabla \boldsymbol{u}, \nabla \varphi) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \varphi) - (p, \nabla \cdot \varphi) = 0, \quad \forall \varphi \in V_1(\Omega), \tag{16b}$$

$$(\chi, \nabla \cdot \boldsymbol{u}) = 0, \quad \forall \chi \in L^2(\Omega). \tag{16c}$$

These formulations too are examined further in later sections. Again it will be seen that these well-posed variational problems contain further boundary conditions along the free boundary S than asked for in (13b). Thus problem (13) by itself is not quite well posed. Figure 6 presents the results of several typical computations for problems with prescribed net fluxes or pressure drops based on the formulations (11) and (15).

The problem of a jet through an aperture in a wall can be regarded as a prototypical problem for computational procedures based on the variational formulations (11) and (15). The computational results shown in Figure 7 are the first of this type that we know of.

(a)

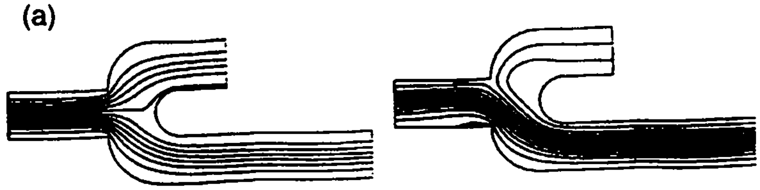

Figure 6(a). Typical steady computations based on the pressure drop formulation (11). Let P_i denote the prescribed mean pressure over the inlet/outlet S_i numbered as in Figure 5. For the computation on the left, $P_1 = 0$, $P_2 = 1{\cdot}5$ and $P_3 = 2$. This produces inflow across S_2 and S_3. For the compuation on the right, $P_1 = 0$, $P_2 = 0{\cdot}5$ and $P_3 = 2$, which produces outflow through both S_1 and S_2. The Reynolds number is approximately 50

(b)

Figure 6(b). Streamlines of flow past an inclined ellipse at $Re = 500$ based on the variational flux formulation (15) with both upstream and downstream boundaries free. Except for the free upstream boundary, all parameters are the same as in Figure 3. Approximately 60% of the flux passes under the ellipse compared with approximately 50% in Figure 3

(c)

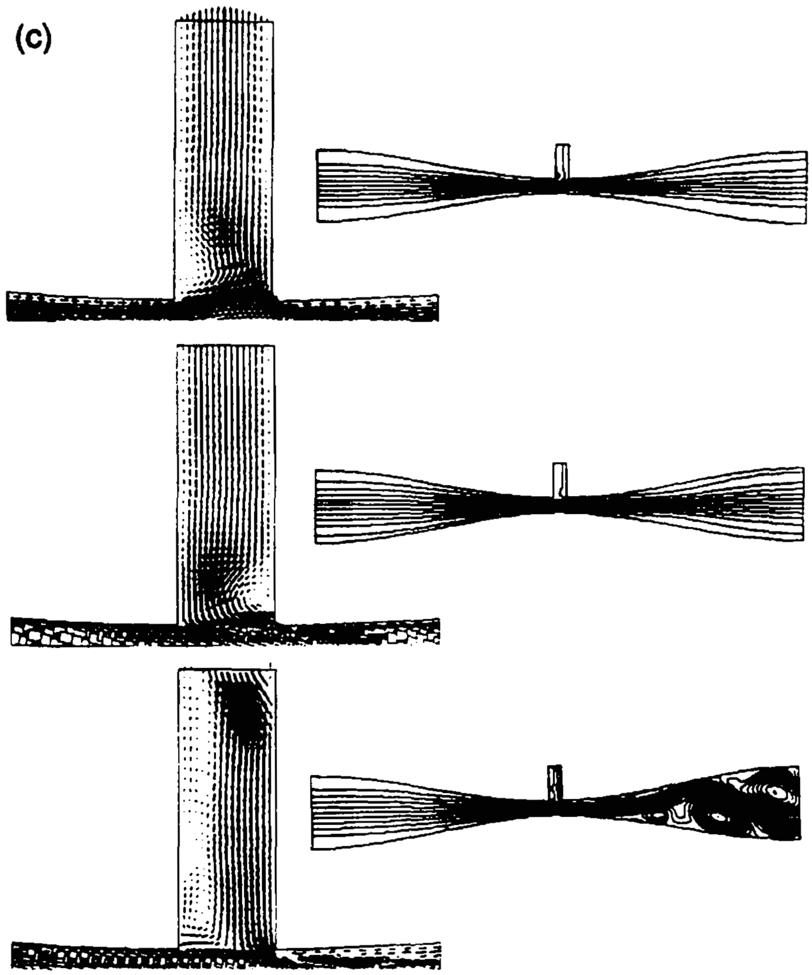

Figure 6(c). Formulations (11) and (15) are combined here. They are combined in an obvious way (introducing a flux carrier from S_1 exiting either S_2 or S_3) in this test of the Bernoulli principle. The results of three computations are shown, with enlargements of the upper duct. In each case an incoming net flux F_1 is prescribed across a free boundary S_1 on the left, while the mean pressure P_2 on a free boundary S_2 at the top of the small upper inlet/outlet is prescribed to be equal to the mean pressure P_3 on a free boundary S_3 at the right. In the first case, at very low Reynolds number, $Re = 10$, there is outflow at S_2. In second case, at $Re = 50$, there is inflow at S_2 as predicted by Bernoulli's principle. In the third case, at $Re = 1000$, there is inflow at S_2 and a complex time-dependent vortex structure in both the upper and downstream ducts. See Figure 6(d) for an enlargement

(d)

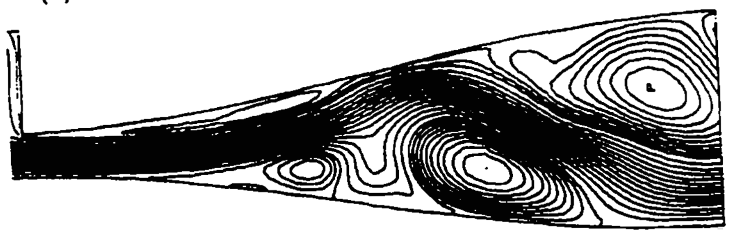

Figure 6(d). Enlargement of part of Figure 6(c)

Before we began our numerical experiments with free inflow boundary conditions, we were concerned that such problems might be quite unstable already on the continuous, theoretical level and that this instability might limit their computational usefulness to very low Reynolds numbers. For instance, it seemed possible that the upstream Dirichlet condition in Figure 3 might be an important stabilizing factor, for lack of which the computations shown in Figures 6(a) and 6(b) would somehow collapse. This concern was heightened by a look at the existence theory for such problems. In Section 6 we present the basic estimates that we know of upon which an existence theory can be given for steady and non-steady solutions of prescribed net flux and pressure drop problems. It will be seen that some of these estimates require assumptions about the smallness of the data that one does not encounter in dealing with Dirichlet boundary conditions, giving the impression (we have not explicitly evaluated the constants) that these theorems may be valid only for very small data. Thus, anticipating difficulties that have not actually arisen in our computations, we looked at alternative variational

(a)

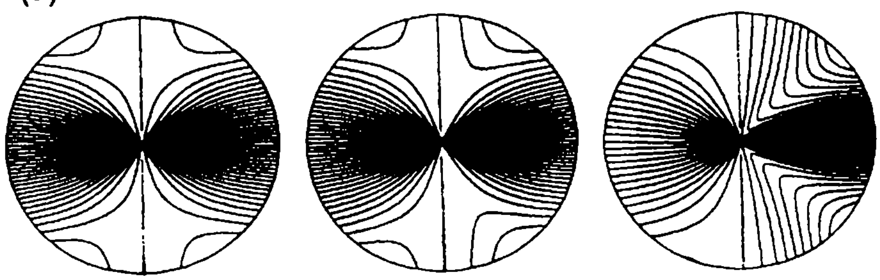

Figure 7(a). Streamlines of a steady jet through an aperture in a wall (a line segment) for $Re = 1$, 10 and 100, based on the variational formulation (15) for flow with a prescribed net flux. The fluid adheres to the linear wall, while the left and right semicircles are free artificial boundaries

(b)

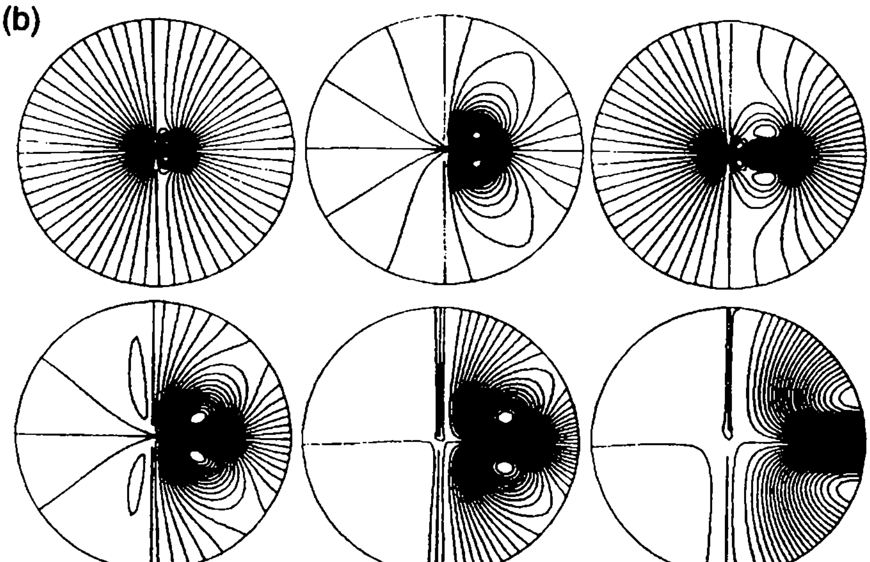

Figure 7(b). Streamlines of a non-steady jet through an aperture in a wall (a line segment, based on the variational formulation (11) for flow with a prescribed time-dependent pressure drop $P(t)$. The initial velocity is $u_0 = 0$ and $P(t)$ is the step function $P(t) = 1$ for $0 \leqslant t \leqslant 40$, $P(t) = 0$ for $41 \leqslant t \leqslant 80$, $P(t) = 1$ for $81 \leqslant t \leqslant 120$ and $P(t) = 0$ for $t \geqslant 121$ (with t in time steps). The figures are for $t = 20$, 60 100, 140, 200 and 500. This produces two short bursts ('puff-puff') through the hole, which are visualized by particle tracing in Figure 7(c)

(c)

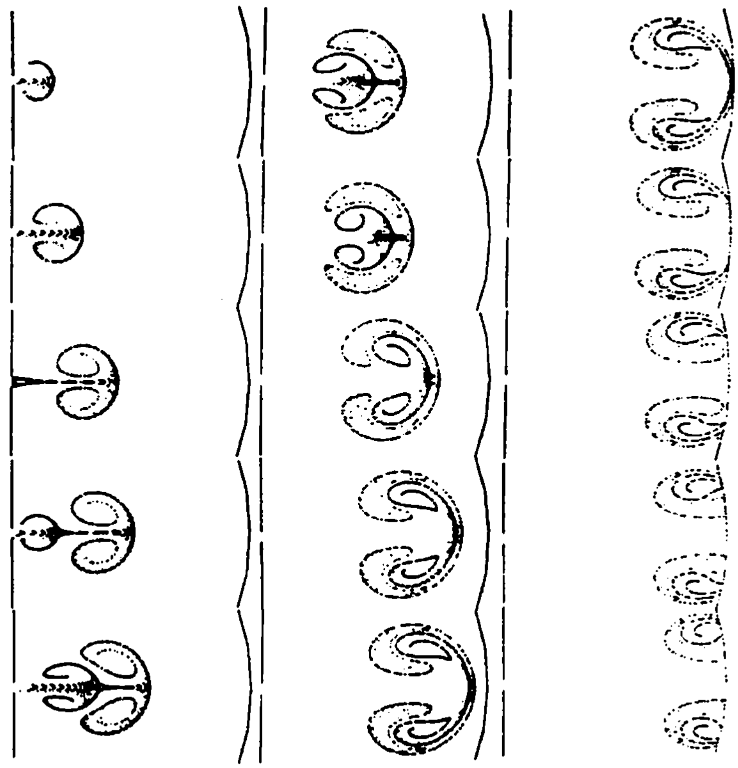

Figure 7(c). The same computations as in Figure 7(b), visualized by tracing particles that are introduced at the aperture during the 'puffs'. The result is two 'smoke rings' that leapfrog each other and eventually exit and the free artificial boundary on the right

formulations of flux and pressure problems using symmetrized 'conservative' forms of the non-linear term. Using these forms, the non-linear term vanishes identically in the energy estimates, facilitating existence theorems for less restrictive data. Of course, changing the variational form also changes the problem that is being solved and may render it unsatisfactory in other respects. That seems to be the case.

One is led to the first of the conservative forms we are referring to by using the identity $\nabla(\frac{1}{2}|\boldsymbol{u}|^2) = \boldsymbol{u}\cdot(\nabla\boldsymbol{u})^{\mathrm{T}}$ to write the Navier–Stokes equations as

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot(\nabla\boldsymbol{u})^{\mathrm{T}} - \nu\Delta\boldsymbol{u} = -\nabla(p + \tfrac{1}{2}|\boldsymbol{u}|^2) = -\nabla\bar{p}. \tag{17}$$

This leads to a variational formulation in which the term $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi)$ is replaced by $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) - (\varphi\cdot\nabla\boldsymbol{u}, \boldsymbol{u})$. On the right side the additional term is absorbed into the pressure, giving what is referred to as 'total pressure' or 'Bernoulli pressure'. The total pressure is constant along streamlines in Euler flow and therefore is an important quantity in some high-Reynolds-number situations. For example, it is the 'Bernoulli principle' that explains the inflow through the central duct of Figure 6(c). The reason that the additional term on the left side of (17) facilitates the existence theory is that when (17) is multiplied through by $\boldsymbol{u}$ to obtain an energy estimate, the non-linear term disappears as if one were considering homogeneous Dirichlet data. Thus motivated, we consider the following alternatives to the problems (11) and (15).

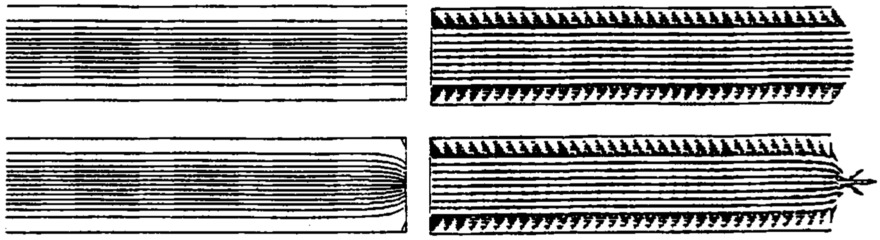

Figure 8. Streamlines and vector plots of pipe flow at $Re = 50$ with an artificial outflow boundary. The upper figures are based on the standard formulation (8) and the lower figures on the total pressure formulation (19)

Variational total pressure drop problem (with solenoidal spaces). Find $\boldsymbol{u}(t)$ such that for all t, $\boldsymbol{u}(t) \in \boldsymbol{J}_1^*(\Omega)$ and

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot(\nabla\boldsymbol{u})^{\mathrm{T}}, \boldsymbol{\varphi}) = -\sum_j P_j(t)\int_{S_j} \boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1^*(\Omega). \tag{18}$$

Variational net flux problem involving total pressure (with solenoidal spaces). Find $\boldsymbol{u}(t) = F_2(t)\mathbf{b}_2 + F_3(t)\mathbf{b}_3 + \boldsymbol{v}(t)$ such that for all t, $\boldsymbol{v}(t) \in \boldsymbol{J}_1(\Omega)$ and

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot(\nabla\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{\varphi}) = 0, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1(\Omega). \tag{19}$$

It will be seen in the next section that the pressure condition corresponding to the problem (18) is no longer (10b) but rather

$$\frac{1}{|S_i|}\int_{S_i}(p + \tfrac{1}{2}|\boldsymbol{u}|^2)\mathrm{d}S = P_i(t). \tag{20}$$

Another conservative form which is often taken for convenience in analysing numerical methods is obtained by replacing $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi)$ by $\frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) - \frac{1}{2}(\boldsymbol{u}\cdot\nabla\varphi, \boldsymbol{u})$ in (11b) and (15b) (see Reference 10, p. 284). This gives a legitimate weak form of the Navier–Stokes equations, because $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) = \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \varphi) - \frac{1}{2}(\boldsymbol{u}\cdot\nabla\varphi, \boldsymbol{u})$ if φ is a solenoidal test function which vanishes on the boundary. When the variational equation (11b) is changed in this way, the new pressure condition corresponding to (10b) is

$$\frac{1}{|S_i|}\int_{S_i}(p + \tfrac{1}{2}|\boldsymbol{u}\cdot\boldsymbol{n}|^2)\mathrm{d}S = P_i(t). \tag{21}$$

Figure 8 presents the result of a typical computation based on the formulation (19).

Clearly the boundary conditions that are implicit in the total pressure variational formulation (19) are not very satisfactory for the problems that we have been considering, although they might perhaps be satisfactory for some other types of problems. To reason further about this, it is necessary to identify the boundary conditions which are implicit in the various formulations that we have been considering.

4. ASSOCIATED BOUNDARY CONDITIONS

It will be shown here that for smooth solutions the variational formulations given above, with and without use of solenoidal spaces, are equivalent and that the prescribed pressure drop problem also admits a formulation in terms of classically prescribed boundary conditions. Solutions of the prescribed flux problem satisfy the same boundary conditions, but with an unknown pressure drop.

Hence the prescribed flux problem does not have a fully equivalent formulation in terms of classical boundary conditions.

Let us consider first the variational pressure drop problems (11) and (12) and show that they are both equivalent to the classical problem (28) below. Referring to the solutions of these three problems as $\boldsymbol{J}_1^*$-, $\boldsymbol{V}_1^*$- and $\boldsymbol{C}^*$-solutions respectively, it is only necessary to verify that $\boldsymbol{V}_1^*$-solutions $\subset \boldsymbol{J}_1^*$-solutions $\subset \boldsymbol{C}^*$-solutions $\subset \boldsymbol{V}_1^*$-solutions. It is obvious that $\boldsymbol{V}_1^*$-solutions $\subset \boldsymbol{J}_1^*$-solutions.

To show that $\boldsymbol{J}_1^*$-solutions are $\boldsymbol{C}^*$-solutions, integrate (11b) by parts to get

$$(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u}, \boldsymbol{\varphi}) + \nu\int_{\partial\Omega}\partial_n\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}S = -\sum_i P_i\int_{S_i}\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1^*(\Omega). \tag{22}$$

Then, using test functions that vanish on the boundary, one may conclude that $\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u}$ is the gradient of a smooth function p which can be defined by curve integrals $p(x) = \int_{x_0}^{x}(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u})\cdot\mathrm{d}\mathbf{s}$ independent of the path. Indeed, the integral $\oint_C(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u})\cdot\mathrm{d}\mathbf{s}$ around any closed curve C in Ω can be approximated by volume integrals $(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u}, \boldsymbol{\varphi})$, where $\boldsymbol{\varphi}$ is a smooth solenoidal function with support confined to a small tube about the curve C, having its streamlines closely aligned with the curve C and carrying a unit net flux in the direction of C. Since these volume integrals vanish by (22), so must the curve integrals and hence $\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\mathbf{u} = -\nabla p$ for some scalar function p. Thus one may set

$$(\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u}, \boldsymbol{\varphi}) = (-\nabla p, \boldsymbol{\varphi}) = -\int_{\partial\Omega}p\boldsymbol{n}\cdot\boldsymbol{\varphi}\,\mathrm{d}S \tag{23}$$

in (22) to get

$$\sum_i\int_{S_i}[\nu\partial_n\boldsymbol{u} + (P_i - p)\boldsymbol{n}]\cdot\boldsymbol{\varphi}\,\mathrm{d}S = 0, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1^*(\Omega). \tag{24}$$

Writing $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ near the boundary in terms of normal and tangential components, i.e. $\boldsymbol{u} = u_\tau\boldsymbol{\tau} + u_n\boldsymbol{n}$, (24) implies

$$\sum_i\int_{S_i}[\nu\partial_n u_n + (P_i - p)]\varphi_n\,\mathrm{d}S = 0, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1^*(\Omega), \tag{25}$$

$$\sum_i\int_{S_i}\partial_n u_\tau\,\varphi_\tau\,\mathrm{d}S = 0, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{J}_1^*(\Omega). \tag{26}$$

Our first conclusion from (25), which is obtained by testing with $\boldsymbol{\varphi}\in\boldsymbol{J}_1(\Omega)$, is that for each S_i there exists a constant c_i such that

$$\nu\partial_n u_n + (P_i - p) = c_i \quad \text{on } S_i. \tag{27}$$

Indeed, if x_1 and x_2 are any two points on S_i, one can connect them by a curve C which is normal to S_i at each of them. Then one can consider flux carriers $\boldsymbol{\varphi}$ with support confined to a small tube about C, with a unit net flux into Ω near x_1 and out of Ω near x_2. Hence, arguing by letting the radius of the tube shrink, (25) implies that $\nu\partial_n u_n + (P_i - p)$ is the same at x_1 and x_2, thus proving (27).

So far we have only been using test functions $\boldsymbol{\varphi}\in\boldsymbol{J}_1(\Omega$. Such functions must satisfy $\int_{S_i}\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S = 0$ on each S_i and therefore cannot carry flux from one outlet to another. However, the full test space $\boldsymbol{J}_1^*(\Omega)$ for (25) also contains flux carriers from one outlet to another, such as the functions $\mathbf{b}_i$ used in formulating the variational prescribed flux problem (15). Using such flux carriers as test functions and arguing as before, we conclude that the constants c_i are all equal to each other. We are free to choose the value of this common constant, because the pressure has so far only been determined up to a constant. We set $c_i = 0$ in (27).

Finally, (26) implies that $\partial_n u_\tau$ vanishes identically at every point of any of the S_i. Indeed, if $x \in S_i$, we can construct a closed curve C in $\bar{\Omega}$ which just grazes the surface S_i at x in any tangential direction. Then, arguing as before, with flux-carrying test functions φ that approximate C, (26) implies that $\partial_n u_\tau = 0$ at x.

What we have shown is that any smooth solution of the variational problem (11) is also a solution of the following.

Classical pressure drop problem. For any prescribed constants P_i find $\boldsymbol{u}(t)$ and $p(t)$ such that for all t

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{28a}$$

$$\boldsymbol{u}|_\Gamma = 0, \qquad (p - \nu \partial_n u_n)|_{S_i} = P_i, \qquad \partial_n u_\tau|_{S_i} = 0. \tag{28b}$$

It will be shown below, following the statement of Theorem 1, that if S_i is a plane section perpendicular to a cylindrical section of pipe as in Figures 2–6, then the boundary condition (28b) implies that P_i is in fact the mean pressure across S_i as originally desired in posing problem (10).

At this point we have shown that $\boldsymbol{J}_1^*$-solutions $\subset$ $\boldsymbol{C}^*$-solutions and it only remains to show that $\boldsymbol{C}^*$-solutions $\subset$ $\boldsymbol{V}_1^*$-solutions. To this end, suppose that $\boldsymbol{u}, p$ satisfies (28). Then it is easily seen that (28b) implies

$$\nu \partial_n \boldsymbol{u} + (P_i - p)\boldsymbol{n} = 0 \quad \text{on each } S_i.$$

Hence, multiplying (28a) by $\varphi \in \boldsymbol{V}_1^*(\Omega)$ and integrating by parts, one obtains

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) &= \int_{\partial\Omega} \nu \partial_n \boldsymbol{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}S - \int_{\partial\Omega} p \boldsymbol{\varphi} \cdot \boldsymbol{n} \, \mathrm{d}S \\ &= -\sum_i P_i \int_{S_i} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, \mathrm{d}S, \end{aligned} \tag{6}$$

which is just (12b). The conditions (12a) and (12c) are obviously satisfied. Thus we have proven the following.

Theorem 1

For smooth solutions, the three formulations (11), (12) and (28) of the prescribed pressure drop problem are all equivalent to each other.

It was our intention in formulating problems (11) and (12) to obtain solutions that satisfy the pressure condition (10b). To check whether this condition is satisfied, we integrate the second of conditions (28b) to obtain for each i

$$\frac{1}{|S_i|} \int_{S_i} p \, \mathrm{d}S = P_i + \frac{\nu}{|S_i|} \int_{S_i} \partial_n u_n \, \mathrm{d}S. \tag{29}$$

The second term on the right can be evaluated using the relation $\nabla \cdot \boldsymbol{u} = 0$. If S_i is a plane section perpendicular to a cylindrical pipe as in Figure 2–6, then this term vanishes identically and (29) reduces to (10b) as desired. However, if S_i is a semicircle as in Figure 7, then the second term on the right side of (29) equals $-\nu F/\pi r^2$, where r is the radius of the semicircle and F is the (unknown) net flux out of Ω across S_i. The corresponding value for a three-dimensional hemisphere is $-\nu F/\pi r^3$. Thus, in calculating flow through a hole in a wall on the basis of the prescribed pressure drop problem, there is a small discrepancy between the pressure drop which was intended and that which is realized, but it decreases rapidly as r is increased.

We make several final remarks concerning the striking success of the boundary conditions (28b) on the artificial boundaries S_i. First, if a straight section of pipe is bounded at its ends by perpendicular sections S_i, then the unique steady solution of (28) is Poiseuille flow. We imagine and intend (perhaps the reader has questioned this) that the domains in Figures 2–6 are truncations of large domains that continue as straight sections of pipe for some distance beyond each of the S_i. Having this intention, any boundary condition which is not satisfied by Poiseuille flow would probably be found unsatisfactory. Second, realizing that no artificial boundary condition can do a perfect job in non-trivial situations, we find it very satisfying that (28b) appears to work so well in calculating flows like those of Figures 6 and 7.

Next we consider the prescribed net flux problem and show that its variational formulations (15) and (16) are equivalent. Referring to the solutions of (15) and (16) as J_1-solutions and V_1-solutions respectively, we first show that V_1-solutions $\subset$ J_1-solutions.

Suppose that a vector field $\boldsymbol{u}(t)$ can be written as $\boldsymbol{u}(t)=F_2(t)\mathbf{b}_2+F_3(t)\mathbf{b}_3+\mathbf{v}(t)$, with $\mathbf{v}(t)$ and the $\mathbf{b}_i$ satisfying the conditions of problem (16). Then $\mathbf{v}(t)\in V_1(\Omega)$ and neither $\mathbf{v}(t)$ nor the $\mathbf{b}_i$ need be solenoidal as required in problem (15). However, choosing solenoidal flux carriers $\tilde{\mathbf{b}}_i$ satisfying, the conditions (14) required in problem (15), we can write $\boldsymbol{u}(t)=F_2(t)\hat{\mathbf{b}}_2+F_3\hat{\mathbf{b}}_2+\tilde{\mathbf{v}}(t)$, where $\tilde{\mathbf{v}}(t)=\mathbf{v}(t)+F_2(t)(\mathbf{b}_2-\tilde{\mathbf{b}}_2)+F_3(t)(\mathbf{b}_3-\tilde{\mathbf{b}}_3)$, and easily check that $\tilde{\mathbf{v}}(t)\in J_1(\Omega)$. Thus a V_1-solution $\boldsymbol{u}(t)$ can be written in the form required of a J_1-solution. Finally, it is obvious that the variational equation (15b) follows from (16b). Thus we find that V_1-solutions are also J_1-solutions.

Now suppose that $\boldsymbol{u}(t)$ is a J_1-solution. Then we can write $\boldsymbol{u}(t)=F_2(t)\mathbf{b}_2+F_3(t)\mathbf{b}_3+\mathbf{v}(t)$, with $\mathbf{v}(t)$ and the $\mathbf{b}_i$ satisfying the conditions of problem (15), which are only stronger than the conditions of problem (16). It is also obvious that (16c) is satisfied. Finally, arguing as we did in going from (22) to (23), we conclude that there exists a scalar function p such that $\boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}=-\nabla p$. Multiplying this by $\boldsymbol{\varphi}\in V_1(\Omega)$ and integrating by parts, we obtain the variational equation (16b). Thus J_1-solutions are also V_1-solutions.

To identify the boundary conditions that are implicit in these problems, it is easiest to consider the V_1-formulation (16). Integrating (16b) by parts, we obtain

$$(\boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla p,\boldsymbol{\varphi})+\int_{\partial\Omega}(\nu\partial_{\boldsymbol{n}}\boldsymbol{u}-p\boldsymbol{n})\cdot\boldsymbol{\varphi}\,\mathrm{d}S=0,\quad\forall\boldsymbol{\varphi}\in V_1(\Omega).\tag{30}$$

Testing with functions $\boldsymbol{\varphi}$ that vanish on the boundary, one sees that $\boldsymbol{u}_t+\boldsymbol{u}\cdot\boldsymbol{u}\cdot\Delta\boldsymbol{u}-\nu\Delta\boldsymbol{u}+\nabla p=0$ and hence that the first integral in (30) vanishes for all $\boldsymbol{\varphi}\in V_1(\Omega)$. Then testing with $\boldsymbol{\varphi}$ that are non-zero on $\partial\Omega$, we easily conclude that

$$\boldsymbol{u}|_\Gamma=0,\qquad(p-\nu\partial_{\boldsymbol{n}}u_{\boldsymbol{n}})|_{S_i}=c_i,\qquad\partial_{\boldsymbol{n}}u_{\boldsymbol{\tau}}|_{S_i}=0\tag{31}$$

for some constants c_i. The argument for this can be made more simply than that for (28b), because the test functions in $V_1(\Omega)$ need not be solenoidal. Notice, however, that since they are constrained by the condition $\int_{S_i}\boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}S=0$, we can only show that $p-\partial_{\boldsymbol{n}}u_{\boldsymbol{n}}$ is constant on each S_i and not necessarily zero. Integrating the second of conditions (31), we get

$$\frac{1}{|S_i|}\int_{S_i}p\,\mathrm{d}S=c_i+\frac{\nu}{|S_i|}\int_{S_i}\int_{S_i}\partial_{\boldsymbol{n}}u_{\boldsymbol{n}}\mathrm{d}S.\tag{32}$$

It is evident that the boundary conditions are the same for the prescribed flux problem as for the prescribed pressure drop problem, except that the mean pressures c_i which appear in them are unknowns. We have proven the following.

Theorem 2

For smooth solutions, the two formulations (15) and (16) of the prescribed net flux problem are equivalent to each other. Their solutions satisfy the same boundary conditions (31) as solutions of the prescribed pressure drop problem, but with mean pressures $c_i(t)$ that are not known in advance of solving the problem.

Using the same methods as above, we obtain the following.

Theorem 3

Smooth solutions of the variational total pressure drop problem (18) and of the variational flux problem involving total pressure, (19), both satisfy the boundary conditions

$$u|_\Gamma = 0, \qquad (p + \tfrac{1}{2}|u|^2 - \nu\partial_n u_n)|_{S_i} = P_i(t), \qquad \partial_n u_\tau = 0 \quad \text{on } S_i. \tag{33}$$

However, for the flux problem the pressures $P_i(t)$ are not known in advance of solving the problem.

Similarly, if one replaces the non-linear term $(u \cdot \nabla u, \varphi)$ in the variational pressure and flux problems (11) and (15) by the symmetrized form $\frac{1}{2}(u \cdot \nabla u, \varphi) - \frac{1}{2}(u \cdot \nabla\varphi, u)$, then the associated boundary conditions are

$$u|_\Gamma = 0, \qquad (p + \tfrac{1}{2}|u_n|^2 - \nu\partial_n u_n)|_{S_i} = P_i(t), \qquad \nu\partial_n u_\tau = \tfrac{1}{2}u_n u_\tau \quad \text{on } S_i. \tag{34}$$

It is evident upon examining the boundary conditions (33) and (34) that they are not satisfied by Poiseuille flow. Thus their poor performance in the computation shown in Figure 8 was to be expected.

Let us briefly consider the variational formulations of problems with artificial boundaries when there are non-zero forces. As a first example, consider flow in a rectangle under the influence of a gravitational force f with the Dirichlet boundary condition $u = 0$ on the entire boundary $\partial\Omega$. Of course, the unique steady solution is $u \equiv 0$ in Ω. Now let us divide the rectangle into left and right halves by an artificial boundary and formulate the Navier–Stokes problem with a gravitational force for the left half alone by adding the term (f, φ) to the right side of (8b) or (11b). The result of a typical computation is shown in Figure 9 and certainly $u \equiv 0$

The reason that we have lost the correct solution $u \equiv 0$ is that the pressure associated with it, $p(x) = \int_{x_0}^{x} f \cdot ds \equiv P_f(x)$, is not constant on S and therefore u and p together do not satisfy (28b). This is easily rectified. In deriving the boundary conditions (28b), we could have considered a general scalar function $P_i(x, t)$ by including it under the integral sign in (11b). The derivation of (28b) remains exactly the same. Thus, to get the correct solution $u \equiv 0, p(x) = P_f(x)$, it is evident that the term $-\int_{\partial\Omega} P_f \varphi \cdot n \, dS$ should be added to the right side of the variational formulations that we have been considering.

In considering more interesting problems such as heat convection modelled by the Boussinesq equations, in which f is essentially the unknown temperature, there is probably no really ideal way of compensating for the variations in pressure along an artificial boundary. However, the simple example just considered suggests the following simple strategy for a partial compensation and shows that nothing simpler can be useful in avoiding the effects demonstrated in Figure 9. One may calculate at every time or periodically a spatially constant mean force $\bar{f} = |\Omega|^{-1} \int_\Omega f \, dx$, from which one gets a simple, linear mean pressure $\bar{P}(x, t) = \int_{x_0}^{x} \bar{f}(t) \cdot ds$ which can be used to define a compensating functional $-\int_{\partial\Omega} \bar{P}\varphi \cdot n \, dS$ for inclusion on the right-hand side of any of the variational formulations that we have considered. Alternatively, one can substract $\nabla\bar{P}(x, t)$ from f in the variational equation. We have not experimented with computations of the Boussinesq equations.

Let us conclude this section with some general remarks about the relationship between variational formulations and their associated boundary conditions. One always starts with a basic function space.

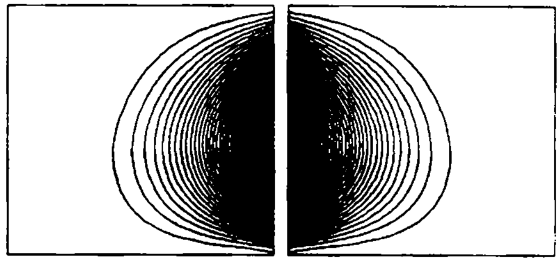

Figure 9. Typical steady state computation restricted to the left half of a rectangle with an artificial boundary and a 'gravitational' force $f=(0, -1)^T$. The Reynolds number is approximately 10

In the solenoidal setting it is $J_1^*(\Omega)$ or a subspace of it defined by further homogeneous constraints such as $\int_{S_i} \varphi \cdot n \, ds = 0, \varphi \cdot n|_{S_i} = 0$ or $\varphi \cdot \tau|_{S_i} = 0$. Let us take first $J_1^*(\Omega)$ itself and consider the problem of finding $u \in J_1^*(\Omega)$ satisfying

$$\nu(Du, D\varphi) + (u_t + u \cdot \nabla u - u \cdot (\nabla u)^T - f, \varphi) = \int_{\partial\Omega} (W + \sigma u) \cdot \varphi \, dS \tag{35}$$

for all $\varphi \in J_1^*(\Omega)$, where $Du = \frac{1}{2}(\nabla u + \nabla u^T)$ is the deformation tensor, $W(x, t)$ is a prescribed force density on $\partial\Omega$ and $\sigma \geqslant 0$ is a prescribed constant. Then, as in deriving (28b), one obtains

$$\int_{\partial\Omega} n \cdot [2Du - (p + \tfrac{1}{2}|u|^2)I] \cdot \varphi \, dS = \int_{\partial\Omega} (W + \sigma u) \cdot \varphi \, dS \tag{36}$$

for all $\varphi \in J_1^*(\Omega)$ and hence the boundary condition (for the effect see Figure 10)

$$2n \cdot Du - (p + \tfrac{1}{2}|u|^2)n = W + \sigma u \quad \text{on each } S_i. \tag{37}$$

Now, if the term $-u \cdot (\nabla u)^T$ is omitted from (35), then the term $\frac{1}{2}|u|^2$ disappears from (36) and (37). If the term $\nu(Du, D\varphi)$ is replaced by $\nu(\nabla u, \nabla\varphi)$ in (35), then the term $(\nabla u)^T$ disappears from (36) and (37). Similarly, if W or σu is omitted from (35), then it disappears in (36) and (37). Next consider the effect of constraining the function space by taking a subspace $\tilde{J}_1(\Omega) \subset J_1^*(\Omega)$ as the basic function space. The first effect is that the constraint is imposed on u, since u is then sought in $\tilde{J}_1(\Omega)$. However, this is balanced by a second freeing effect due to there being fewer test functions available in drawing conclusions from (36). Thus the constraint $\int_{S_i} \varphi \cdot n \, ds = 0$ has the effect of introducing an unknown constant $c_i n$ into (37), as we saw above in considering the net flux problem. The constraint $\varphi \cdot n|_{S_i} = 0$ loses the normal component of the boundary condition (37) altogether. The constraint $\varphi \cdot \tau|_{S_i} = 0$ loses the tangential constraint of the boundary condition (37). There are many interesting possibilities; see e.g. Reference 11, Chap. 4.6.

Figure 10. Vector plots of pipe flow at $Re = 50$ with an artificial outflow boundary. The computation on the left is based on the standard variational formulation (8). The computation on the right is made similarly but with the term $\nu(\nabla u, \nabla\varphi)$ in (8) replaced by $\nu(Du, D\varphi)$, where D is the deformation tensor

5. STREAMFUNCTION FORMULATIONS

It is often useful to formulate two-dimensional problems in terms of a streamfunction. The pressure drop and net flux problems are very conveniently posed in this way. This raises the question of whether solutions obtained using the natural 'do nothing' inflow/outflow boundary conditions in the streamfunction formulation really coincide with those of the corresponding formulations in the primal variables. The following analysis shows that they do.

In two dimensions the portion of the boundary denoted by Γ in Figure 5 is the union of three disconnected components, $\Gamma = \cup\Gamma_i$, and the full boundary $\partial\Omega$ can be oriented in the counterclockwise direction by a tangent vector $\boldsymbol{\tau}$ as shown in Figure 11.

We denote the values of a scalar-valued function ϕ at the two ends of each outlet S_i, oriented as mentioned, by ϕ_{i1} and ϕ_{i2}. Also, $\mathbf{curl}\phi \equiv (\partial_2\phi, -\partial_1\phi)^{\mathrm{T}}$. Below, the solutions of the prescribed pressure drop and prescribed net flux problems are sought in the form $\boldsymbol{u} = \mathbf{curl}\psi$. Since ψ is constant along streamlines in such a representation, it is referred to as a streamfunction.

If C is any smooth curve in $\bar{\Omega}$ going from a first point x_1 to a second point x_2 and if $\boldsymbol{\tau}$ is the unit tangent vector to C oriented in the forward-looking direction and $\boldsymbol{n}$ is the right-side normal $\boldsymbol{n} = (\tau_2, -\tau_1)$, then the net flux of $\mathbf{curl}\phi$ crossing C from left to right between x_1 and x_2 is

$$\int_C \mathbf{curl}\phi \cdot \boldsymbol{n}\, ds = \int_C (n_1\partial_2\phi - n_2\partial_1\phi) dS = \int_C \partial_\tau\phi\, dS = \phi(x_2) - \phi(x_1). \tag{38}$$

Consequently, if $\boldsymbol{\varphi} = \mathbf{curl}\phi$, the right side of (11b) can be expressed as $-\sum_i P_i(\phi_{i2} - \phi_{i1})$. Below we show that the pressure drop problem (11) is equivalent to the following.

Variational pressure drop problem (streamfunction formulation): Find $\psi(t)$ such that for all t

$$\psi(t) \in \overset{*}{H}_2(\Omega) \equiv \{\phi \in W_2^2(\Omega) : \partial_n\phi|_\Gamma = 0, \phi|_{\Gamma_i} = c_i, \forall i\}, \tag{39a}$$

$$\nu(\nabla\mathbf{curl}\psi, \nabla\mathbf{curl}\phi) + (\mathbf{curl}\psi_t + \mathbf{curl}\psi \cdot \nabla\mathbf{curl}\psi, \mathbf{curl}\phi) = -\sum_i P_i(\phi_{i2} - \phi_{i1}) \tag{39b}$$

for every $\phi \in \overset{*}{H}_2(\Omega)$.

The c_i in the definition of $\overset{*}{H}_2(\Omega)$ are arbitrary constants. However, one of them can be fixed without losing generality in the vector fields $\mathbf{curl}\phi$. Requiring one of them to be zero, say $c_1 = 0$, has the advantage of making $\|\nabla^2\phi\|_{L^2(\Omega)}$ a norm on $\overset{*}{H}_2(\Omega)$. Below, we will show that $\mathbf{curl}\overset{*}{H}_2(\Omega) = \overset{*}{\boldsymbol{J}}_1(\Omega)$, from which it is obvious that the formulations (11) and (39) are equivalent.

To formulate a streamfunction equivalent of the prescribed net flux problem (15), we need to introduce streamfunction analogues of the flux carriers $\mathbf{b}_i$. Let us assume that there are three inlet/outlets. Then, referring to Figure 11, we take $\psi_2, \psi_3 \in \overset{*}{H}_2(\Omega)$ satisfying

$$\psi_2|_{\Gamma_1} = 1, \quad \psi_2|_{\Gamma_2\cup\Gamma_3} = 0 \quad \text{and} \quad \psi_3|_{\Gamma_1\cup\Gamma_2} = 1, \quad \psi_3|_{\Gamma_3} = 0. \tag{40}$$

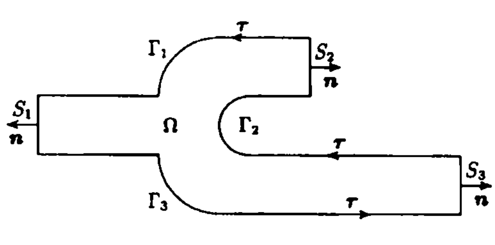

Figure 11. Notation for flow regions having artificial boundaries at multiple inlet/outlets S_i

It is easily seen using (38) that we obtain flux carriers $\mathbf{b}_i$ satisfying (14) by letting $\mathbf{b}_i = \mathbf{curl}\psi_i$ $(i = 2, 3)$. Thus we may seek $\boldsymbol{u}$ in the form $\boldsymbol{u} = \mathbf{curl}\psi$ with streamfunction ψ in the form $\psi = F_2\psi_2 + F_3\psi_3 + \eta$, where $\mathbf{curl}\eta$ has zero net flux across each S_i.

Variational net flux problem (streamfunction formulation). Find $\psi(t) = F_2(t)\psi_2 + F_3(t)\psi_3 + \eta(t)$ such that for all t

$$\eta(t) \in H_2(\Omega) \equiv \{\phi \in W_2^2(\Omega) : \partial_n \phi|_\Gamma = 0, \phi|_\Gamma = 0\}, \tag{41a}$$

$$\nu(\nabla\mathbf{curl}\psi, \nabla\mathbf{curl}\phi) + (\mathbf{curl}\psi_t + \mathbf{curl}\psi \cdot \nabla\mathbf{curl}\psi, \mathbf{curl}\phi) = 0 \tag{41b}$$

for every $\phi \in H_2(\Omega)$.

It is obvious that the problems (41) and (15) are equivalent if $\mathbf{curl}H_2(\Omega) = \boldsymbol{J}_1(\Omega)$.

Theorem 4

$\mathbf{curl}H_2^*(\Omega) = \boldsymbol{J}_1^*(\Omega)$ and $\mathbf{curl}H_2(\Omega) = \boldsymbol{J}_1(\Omega)$. Therefore the two-dimensional prescribed pressure drop problems (39) and (11) are equivalent and so also are the prescribed net flux problems (41) and (15).

First, it is clear that $\mathbf{curl}H_2^* \subset \boldsymbol{J}_1^*(\Omega)$. To prove the reverse, let φ be a given element of $\boldsymbol{J}_1^*(\Omega)$ which we assume at first to be smooth. In seeking a function $\phi \in H_2^*(\Omega)$ such that $\mathbf{curl}\phi = \varphi$, we are motivated by (38) to choose an arbitrary fixed point $x_0 \in \bar{\Omega}$ and set

$$\phi(x) = \int_C \varphi \cdot \boldsymbol{n}\, \mathrm{dS}, \tag{42}$$

where C is an arbitrary piecewise smooth curve in $\bar{\Omega}$ from x_0 to x. It is clear that $\phi(x)$ is independent of the choice of C, since $\nabla \cdot \varphi = 0$. It is also easily checked that $\mathbf{curl}\phi = \varphi$. Finally, $\phi \in H_2^*(\Omega)$, since $\nabla\phi = 0$ on each component of Γ.

Now, dropping the smoothness assumption on φ, suppose that $\|\nabla(\varphi_k - \varphi)\|_{L^2(\Omega)} \to 0$, where the φ_k are smooth elements of $\boldsymbol{J}_1^*(\Omega)$. Constructing ϕ_k from φ_k by the formula (42), we have $\phi_k \in H_2^*(\Omega)$ and $\mathbf{curl}\phi_k = \varphi_k$. Now $\{\nabla\mathbf{curl}\phi_k\}$ is a Cauchy sequence in $\boldsymbol{L}^2(\Omega)$ with limit $\nabla\varphi$. Therefore ϕ_k is a Cauchy sequence in $H_2^*(\Omega)$ with limit ϕ satisfying $\mathbf{curl}\phi = \varphi$ provided that $\|\nabla\mathbf{curl}\cdot\|_{L^2(\Omega)}$ is a norm in $H_2^*(\Omega)$. As mentioned following the introduction of $H_2^*(\Omega)$ in (39), we may specify that $c_1 = 0$. Making that specification here, we obtain $\|\phi\|_{W_2^2(\Omega)} \leqslant c\|\nabla^2\phi\|_{L^2(\Omega)}$ for $\phi \in H_2^*(\Omega)$ by Poincaré's inequality and then $\|\nabla^2\phi\|_{L^2(\Omega)} = \|\nabla\mathbf{curl}\phi\|_{L^2(\Omega)}$ by the identity $\nabla\mathbf{curl}\phi : \nabla\mathbf{curl}\psi = \sum_{ij} \partial_i\partial_j\phi\partial_i\partial_j\psi$. This completes the proof that $\mathbf{curl}H_2^*(\Omega) = \boldsymbol{J}_1^*(\Omega)$.

Finally, we see that $\mathbf{curl}H_2(\Omega) \subset \boldsymbol{J}_1(\Omega)$, because the condition $\phi|_\Gamma = 0$ combined with (38) implies $\int_{S_i} \mathbf{curl}\phi \cdot \boldsymbol{n}\, \mathrm{d}s = 0$. Also, we see that $\boldsymbol{J}_1(\Omega) \subset \mathbf{curl}H_2(\Omega)$, because given $\varphi \in \boldsymbol{J}_1(\Omega)$ and defining ϕ by (42) with $x_0 \in \Gamma$, the conditions $\int_{S_i} \varphi \cdot \boldsymbol{n}\, \mathrm{d}s = 0$ combined with (38) imply $\phi|_\Gamma = 0$. This completes the proof of Theorem 4.

Since the problems (39) and (41) are equivalent to the problems (11) and (15) respectively, the corresponding natural boundary conditions satisfied by smooth solutions along the inlet/outlets must coincide. This, however, leads to something of a puzzle. The primal solution $\{\boldsymbol{u}, p\}$, say of the pressure drop problem (11), satisfies the first-order natural boundary conditions (28b),

$$\partial_n u_\tau|_{S_i} = 0, \qquad (p - \nu\partial_n u_n)|_{S_i} = P_i. \tag{43}$$

On the other hand, the corresponding streamfunction solution ψ is determined through a fourth-order problem, similar to a plate-bending problem with part of the boundary left free. It therefore necessarily satisfies two natural boundary conditions along the inlet/outlets, one of second order and one of third order. This third-order boundary condition for ψ would seem to result in a second-order boundary condition for $\boldsymbol{u} = \mathbf{curl}\psi$, in contrast with what are only first-order conditions in (28b). This apparent paradox is resolved as follows. First, integrating by parts in the variational equation (39b) and varying the test function in $H_2^*(\Omega)$, one obtains in the usual way the necessary condition

$$\Delta^2\psi - \Delta\psi_t - \mathbf{curl}(\mathbf{curl}\psi \cdot \mathbf{curl}\psi) = 0 \quad \text{in } \Omega. \tag{44}$$

The remaining boundary integral takes the form

$$\nu \int_{\partial\Omega} (A(\psi)\phi + \partial_n^2\psi\partial_n\phi + \partial_n\partial_\tau\psi\partial_\tau\phi)\mathrm{d}S = -\sum_i P_i \int_{S_i} \partial_\tau\phi \, \mathrm{d}S, \tag{45}$$

where $A(\psi) = -\partial_n\Delta\psi + \partial_n\psi_t + \boldsymbol{\tau}\cdot(\mathbf{curl}\psi \cdot \nabla\mathbf{curl}\phi)$. For a smooth solution $\boldsymbol{u} = \mathbf{curl}\psi$ there holds the identity

$$-\nu\mathbf{curl}\Delta\psi + \mathbf{curl}\psi_t + \mathbf{curl}\psi \cdot \nabla\mathbf{curl}\psi = -\nu\Delta\boldsymbol{u} + \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p, \tag{46}$$

yielding $-\nu\partial_n\Delta\psi + \partial_n\psi_t + \boldsymbol{\tau}\cdot(\mathbf{curl}\psi \cdot \nabla\psi) = -\partial\tau p$ on $\partial\Omega$. Thus integrating by parts over $\partial\Omega$ yields

$$\sum_i \int_{S_i} [\nu\partial_n^2\psi\partial_n\phi + (\nu\partial_n\partial_\tau\psi - p + P_i)\partial_\tau\phi]\mathrm{d}s = 0, \quad \forall\phi \in H_2^*(\Omega). \tag{47}$$

This implies that $\partial_n^2\psi|_{S_i} = 0$ and $(\nu\partial_n\partial_\tau\psi - p)|_{S_i} = -P_i$, which is just the boundary condition (43) expressed in terms of the streamfunction ψ.

In Theorems 1 and 2 we decided not to burden the reader with explicit constructions of the solenoidal test functions, since the constructions are somewhat technical in three dimensions. In two dimensions, however, they are easily constructed using streamfunctions. According to (38), the net flux of $\mathbf{curl}\phi$ across any curve joining two points x_1 and x_2 is just $\phi(x_2) - \phi(x_1)$. Thus one obtains flux carriers $\mathbf{curl}\phi_\varepsilon$ around a closed curve C by simply mollifying a step function ϕ that equals zero inside C and unity outside C. Similarly, if C is a curve joining points on S_i and S_j ($i = j$ or $i \neq j$), one gets a flux carrier from S_i to S_j by mollifying a step function which equals zero on one side of C and unity on the other side.

6. EXISTENCE, UNIQUENESS, CONTINUOUS DEPENDENCE, STABILITY

We will give a brief account of the existence theory that we see for the problems that have been considered in this paper. It is less complete than for Dirichlet boundary conditions because of a difficulty in estimating the energy that enters the domain across the boundary when there is an inflow.

This difficulty is avoided, however, if one uses the conservative forms of the non-linear term which were discussed at the ends of Sections 3 and 4. Then the existence theory proceeds almost exactly as for Dirichlet conditions. With a few seemingly appropriate restrictions on the domain, one gets smooth steady solutions (but without any assurance of their stability if the data are large) for any prescriptions of steady net fluxes F_i or pressures P_i. For suitably smooth initial values and time-dependent fluxes $F_i(t)$ or pressures $P_i(t)$, regardless of their size, one gets a global (i.e. existing for all $t \geqslant 0$) weak solution which is smooth on an initial time interval $0 < t < T$. In the case of two dimensions, $T = \infty$. In the case of three dimensions, $T = \infty$ if the data are sufficiently small. Since the proofs differ very little from those for Dirichlet conditions, there is no need to present them here.

Turning to existence questions connected with the standard form of the non-linear term used in problems (11) and (15) and their equivalents, our results differ from those above in the following ways. First, we have been unable to obtain an *a priori* bound for the Dirichlet norms of steady solutions even when the data are small. The technique of Leray and Hopf for bounding the Dirichlet norm in the case of non-homogeneous Dirichlet data is of no avail even for the prescribed net flux problem, because it does not apply to the most troublesome term. Despite that, we are able to prove the existence of smooth steady solutions with bounded Dirichlet norms in the case of small data (the prescribed F_i or P_i). One gets an impression from the proof, however, that the data may have to be very small. For the non-stationary problems we get, as before, the existence of a smooth solution on an initial time interval $0 < t < T$, with $T = \infty$ if the data are sufficiently small. However, if the data are large, we have not proven the global existence of even a weak solution, even in two dimensions, as there seems to be a basic difficulty in getting a global energy estimate.

To prove an existence theorem for a Navier–Stokes problem, either steady or non-steady, it is convenient to construct the solution as a limit of Galerkin approximations in terms of the eigenfunctions of the corresponding steady Stokes problem. This use of the Stokes eigenfunctions originated with Prodi[12] and was further developed by Heywood[13] and Heywood and Rannacher.[14] Reference 13 and a related work on Burgers' equation by Heywood and Xie[15] would probably be most useful to a reader who seeks help in the details of what follows.

Let us consider first problems with prescribed pressure drops. To define the corresponding Stokes operator, we introduce $\boldsymbol{J}^*(\Omega)$ as the completion of $\boldsymbol{J}_1^*(\Omega)$ in $\boldsymbol{L}^2(\Omega)$. Then for every $\boldsymbol{f} \in \boldsymbol{J}^*(\Omega)$ there exists exactly one $\boldsymbol{w} \in \boldsymbol{J}_1^*(\Omega)$ such that

$$(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{J}_1^*(\Omega). \tag{48}$$

Moreover, for each $\boldsymbol{w} \in \boldsymbol{J}_1^*(\Omega)$ there is at most one $\boldsymbol{f} \in \boldsymbol{J}^*(\Omega)$ satisfying (48). Thus (48) defines a one-to-one correspondence between functions $\boldsymbol{f} \in \boldsymbol{J}^*(\Omega)$ and functions $\boldsymbol{w}$ in a subspace of $\boldsymbol{J}_1^*(\Omega)$ that we denote by $\boldsymbol{D}(\tilde{\Delta})$. Writing $\tilde{\Delta}\boldsymbol{w} = -\boldsymbol{f}$ defines the desired Stokes operator $\tilde{\Delta} : \boldsymbol{D}(\tilde{\Delta}) \to \boldsymbol{J}^*(\Omega)$. Its inverse $\tilde{\Delta}^{-1}$ is completely continuous and self-adjoint as a mapping $\tilde{\Delta}^{-1} : \boldsymbol{J}^*(\Omega) \to \boldsymbol{J}^*(\Omega)$. Therefore it posseses a sequence of eigenfunctions $\{\boldsymbol{a}^k\}$, which are complete and orthogonal in both $\boldsymbol{J}^*(\Omega)$ and $\boldsymbol{J}_1^*(\Omega)$.

Below, we assume that the inequalities ($\|\cdot\|$ denoting the norm of $\boldsymbol{L}^2(\Omega)$)

$$\sup_\Omega |\boldsymbol{w}| \leqslant \begin{cases} c_0 \|\boldsymbol{w}\|^{1/2} \|\tilde{\Delta}\boldsymbol{w}\|^{1/2} & \text{if } n = 2, \\ c_1 \|\nabla \boldsymbol{w}\|^{1/2} \|\tilde{\Delta}\boldsymbol{w}\|^{1/2} & \text{if } n = 2 \text{ or } 3, \end{cases}$$

$$\|\nabla \boldsymbol{w}\| \leqslant c_2 \|\tilde{\Delta}\boldsymbol{w}\| \quad \text{if } n = 2 \text{ or } 3$$

are valid for every $\boldsymbol{w} \in \boldsymbol{D}(\tilde{\Delta})$. They may be valid for arbitrary bounded two- or three-dimensional domains in analogy with a recent result for the Laplacian by Xie.[16] However, to date, the only known proofs of (49) in the situation of Figure 5 require that Γ and the S_i are smooth and meet at right angles (or nearly so) and that the domain is two-dimensional: see e.g. Reference 17.

Galerkin approximations $\boldsymbol{u}^m = \sum_{k=1}^m c_{km}(t)\boldsymbol{a}^k$ are defined for the pressure drop problem (11) as solutions of the finite system of equations (for simplicity we denote $\boldsymbol{u}^m$ by $\boldsymbol{u}$)

$$(\boldsymbol{u}_t, \boldsymbol{\varphi}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) = -(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) - \sum_i P_i \int_{S_i} \boldsymbol{\varphi} \cdot \boldsymbol{n} \, dS, \quad \forall \boldsymbol{\varphi} \in \text{span}\{a^1, \ldots, a^m\}. \tag{50}$$

In seeking steady solutions, $\boldsymbol{u}_t = 0$ and (50) is a system of algebraic equations for constant unknowns c_{km}. In seeking non-stationary solutions, (50) is a system of ordinary differential equations and its solutions are required to satisfy the initial conditions $(\boldsymbol{u}(0) - \boldsymbol{u}_0, \boldsymbol{\varphi}) = 0$, $\forall \boldsymbol{\varphi} \in \text{span}\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^m\}$, where $\boldsymbol{u}_0$ is the prescribed initial velocity. The underlying estimates for the existence theorems below are obtained by setting $\boldsymbol{\varphi} = \boldsymbol{u}$ or $\boldsymbol{\varphi} = -\tilde{\Delta}\boldsymbol{u}$ in (50). The latter is possible because $\tilde{\Delta}\boldsymbol{u} \in \text{span}\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^m\}$, since $\{\boldsymbol{a}^k\}$ is a spectral basis.

Let us consider first the steady problem. Setting $\varphi = \boldsymbol{u}$ in (50) and estimating the terms on the right side very crudely by

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{u})| \leqslant \|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}\|_{L^3} \leqslant c_3\|\nabla\boldsymbol{u}\|^3, \quad n = 2 \text{ or } 3, \tag{51}$$

$$\left|\sum_i P_i \int_{S_i} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S\right| \leqslant c_4 P\|\boldsymbol{u}\|, \tag{52}$$

where the constant c_3 depends on Sobolev's and Poincaré's inequalities and c_4 depends on a trace inequality, we obtain from (50) the inequality

$$\nu\|\nabla\boldsymbol{u}\| \le c_3\|\nabla\boldsymbol{u}\|^2 + c_4 P. \tag{53}$$

This limits $\|\nabla\boldsymbol{u}\|$ to the right side of the parabola in Figure 12.

The estimate (53) shown in Figure 12 suggests the following theorem.

Theorem 5

For $P \equiv \sum_i |P_i| \leqslant \nu^2/4c_3c_4$ there exists a steady smooth solution of the variational problem (11), and of its equivalents (12) and (39), satisfying

$$\|\nabla\boldsymbol{u}\| \leqslant \frac{\nu}{2c_3}\left[1 - \sqrt{\left(1 - \frac{4c_3c_4P}{\nu^2}\right)}\right]. \tag{54}$$

The main points to be shown in proving this are first that the algebraic equations (50) have solutions $\boldsymbol{u}^n$ and then that the solutions $\boldsymbol{u}^n$ satisfy the estimate (54).

To prove the solvability of the finite-dimensional problems (50), we use Brouwer's fixed point theorem, applying it to the mapping $\boldsymbol{w}\to\boldsymbol{u}$ defined by the linear problem

$$\nu(\nabla\boldsymbol{u},\nabla\varphi) + (\boldsymbol{w}\cdot\nabla\boldsymbol{u},\varphi) = -\sum_i P_i\int_{S_i}\varphi\cdot\boldsymbol{n}\,\mathrm{d}S, \quad \forall\varphi \in \operatorname{span}\{a^1,\ldots,\boldsymbol{a}^m\}, \tag{55}$$

where for brevity the superscript m has again been dropped, $\boldsymbol{u}=\boldsymbol{u}^m$. These linear equations are uniquely solvable if $\boldsymbol{w}$ lies in the ball (54), because then $\boldsymbol{u}=0$ is the only solution of the corresponding homogeneous equation ($P_i=0$). Indeed, if $\boldsymbol{w}$ satisfies (54) and $\boldsymbol{u}$ satisfies (55) with the $P_i=0$, then

$$\nu\|\nabla\boldsymbol{u}\|^2 \leqslant c_3\|\nabla\boldsymbol{w}\|\|\nabla\boldsymbol{u}\|^2 \leqslant c_3\frac{\nu}{2c_3}\|\nabla\boldsymbol{u}\|^2,$$

which implies that $\boldsymbol{u}=0$. To see that the mapping $\boldsymbol{w}\to\boldsymbol{u}$ takes the ball defined by (54) into itself, suppose that $\boldsymbol{w}$ satisfies (54). Then, similarly to (53), we obtain

$$\nu\|\nabla\boldsymbol{u}\| \leqslant c_3\|\nabla\boldsymbol{w}\|\|\nabla\boldsymbol{u}\|^2 + c_4P$$

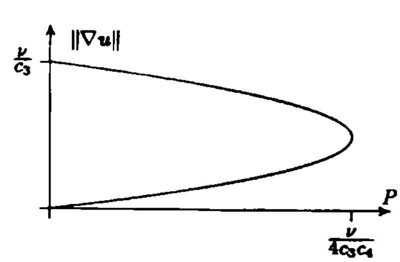

Figure 12. If $\boldsymbol{u}$ is a steady solution of the prescribed mean pressure problem (formulation (11), (12) or (39)) with $P < \nu/4c_3c_4$, then its Dirichlet norm $\|\nabla\boldsymbol{u}\|$ must have a value above the upper branch of the parabola or below the lower branch. Theorem 5 gives the existence of a solution with $\|\nabla\boldsymbol{u}\|$ below the lower branch

and therefore

$$\|\nabla \boldsymbol{u}\| \leqslant \frac{c_4 P}{\nu - c_3\|\nabla \boldsymbol{w}\|} \leqslant \frac{c_4 P}{\nu - (\nu/2)[1 - \sqrt{(1 - 4c_1c_2P/\nu^2)}]} = \frac{\nu}{2c_3}\left[1 - \sqrt{\left(1 - \frac{4c_3c_4P}{\nu^2}\right)}\right].$$

Thus Brouwer's fixed point theorem can be applied and gives the existence of Galerkin approximations satisfying (54). Hence by a standard compactness argument there is at least a subsequence of the Galerkin approximations coverging to a weak solution $\boldsymbol{u} \in J_1^*(\Omega)$ of the steady problem (11). Its smoothness is easily proven if one obtains a further estimate from the Galerkin approximations by setting $\varphi = -\tilde{\Delta}\boldsymbol{u}$ in (50). This gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|^2 + \nu\|\tilde{\Delta}\boldsymbol{u}\|^2 = (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \tilde{\Delta}\boldsymbol{u}) + \sum_i P_i \int_{S_i} \tilde{\Delta}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{56}$$

Because $\tilde{\Delta}\boldsymbol{u}$ is solenoidal, one has the rather unusual trace estimate

$$\left|\sum_i P_i \int_{S_i} \tilde{\Delta}\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S\right| \leqslant c_5 P\|\tilde{\Delta}\boldsymbol{u}\|, \tag{57}$$

which we combine with (49) and (56) to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|^2 + \nu\|\tilde{\Delta}\boldsymbol{u}\|^2 \leqslant c_1\|\nabla\boldsymbol{u}\|^{3/2}\|\tilde{\Delta}\boldsymbol{u}\|^{3/2} + c_5P\|\tilde{\Delta}\boldsymbol{u}\|. \tag{58}$$

Then, using Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|^2 + \nu\|\tilde{\Delta}\boldsymbol{u}\|^2 \leqslant \frac{4c_1^4}{\nu^3}\|\nabla\boldsymbol{u}\|^6 + \frac{4c_5^2}{\nu}P^2. \tag{59}$$

In the steady case this yields an estimate for the Galerkin approximations of the form

$$\|\tilde{\Delta}\boldsymbol{u}\| \leqslant c_\nu\|\nabla\boldsymbol{u}\|^3 + c_\nu P, \tag{60}$$

which is then inherited by the solution. The full classical smoothness of the solution can now be obtained using the L^2-regularity theory for the steady Stokes equations.[13] This completes the proof of Theorem 5. For the non-steady problem we have the following results.

Theorem 6

For any smooth $P_i(t)$ and any prescribed initial value $\boldsymbol{u}_0 \in J_1^*(\Omega)$ there exists a positive number T and a unique smooth solution $\boldsymbol{u}$ of the non-steady problem (11), and of its equivalents (12) and (39), which is defined on at least the initial time interval $0 \leqslant t < T$ and satisfies $\boldsymbol{u}(0) = \boldsymbol{u}_0$. The solution exists for all $t \geqslant 0$ if $P \equiv \sup_{t \geqslant 0} \sum |P_i(t)|$ and $\|\nabla\boldsymbol{u}_0\|$ are sufficiently small. It is also exponentially stable if these quantities are small enough. These results are all valid in both two and three dimensions.

The basic estimates for this theorem are obtained by setting $\varphi = -\tilde{\Delta}\boldsymbol{u}$ in (50) and proceeding as above to obtain (58) and (59). It is clear that (59) can be integrated on some interval $0 \leqslant t < T$. Hence the proof of the existence of a solution with full classical regularity can be completed by the methods of References 13–15. Uniqueness is proved below. This completes the proof of the first part of the theorem.

To prove global existence for small data, first use the third of the inequalities (49) together with (58) to get

$$\frac{d}{dt}\|\nabla u\|^2 + (\nu - 2c_1c_2^{1/2}\|\nabla u\|)\|\tilde{\Delta} u\|^2 \leqslant \frac{c_5^2}{\nu}P^2. \tag{61}$$

Then, on any time interval during which $\|\nabla u\| \leqslant \nu/4c_1c_2^{1/2}$, (61) and (49) imply

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{\nu}{2}\|\nabla u\|^2 \leq \frac{c_5^2}{\nu}P^2. \tag{62}$$

This is easily seen to imply that $\|\nabla u(t)\| \leqslant \nu/4c_1c_2^{1/2}$ for all $t \geqslant 0$ if $\|\nabla u_0\| \leqslant \nu/4c_1c_2^{1/2}$ and $P < \nu^2/\sqrt{(32c_2)}c_1c_5$.

The inequalities (51)–(62) are valid for both two and three dimensions. They do not use the full power of the two-dimensional inequality (49). Using that, one obtains in place of (59) the two-dimensional estimate

$$\frac{d}{dt}\|\nabla u\|^2 + \nu\|\tilde{\Delta} u\|^2 \leqslant \frac{4c_0^2}{\nu^3}\|u\|^2\|\nabla u\|^4 + \frac{4c_5^2}{\nu}P^2. \tag{63}$$

This can be integrated for all time, regardless of the size of the data, provided that $\|u(t)\|$ and $\int_0^t \|\nabla u(s)\|^2\, ds$ remain finite. The natural way to attempt to show that these quantities remain finite is to set $\varphi = u$ in (50), getting the energy identity

$$\frac{1}{2}\frac{d}{dt}\|u\| + \nu\|\nabla u\|^2 = -(u\cdot\nabla u, u) - \sum_i P_i\int_{S_i} u\cdot n\, dS, \tag{64}$$

and then estimate the terms on its right side. The trace inequality

$$\int_{\partial\Omega}(u\cdot n)^2 dS \leqslant c_6\|u\|^2 \tag{65}$$

for solenoidal functions can be used to sharpen the estimate (51), as follows:

$$\begin{aligned}|(u\cdot\nabla u, u)| &= \left|\frac{1}{2}\int_{\partial\Omega}(u\cdot n)u^2\, dS\right| \leqslant \frac{1}{2}\left(\int_{\partial\Omega} u_n^2\, dS\right)^{1/2}\left(\int_{\partial\Omega} u^4\, dS\right)^{1/2}\\ &\leq \begin{cases} c_7\|u\|^{5/4}\|\nabla u\|^{7/4} & \text{if } n = 2,\\ c_8\|u\|\,\|\nabla u\|^2 & \text{if } n = 2 \text{ or } 3.\end{cases}\end{aligned} \tag{66}$$

However, even the two-dimensional version of (66) combined with (64) yields only

$$\frac{d}{dt}\|u\|^2 + \nu\|\nabla u\|^2 \leqslant \frac{32c_7}{\nu^7}\|u\|^{10} + 2c_5P\|u\| \tag{67}$$

and hence only a local energy estimate for large data. It leads neither to a qualitative improvement in Theorem 6 nor to a large-data global existence theorem for weak, solutions even, in two dimensions.

The inequality (66) does facilitate the treatment of uniqueness, continuous dependence on the initial data and simple energy stability. If $w = v - u$ is the difference of two solutions of (11) or of (50), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu\|\nabla w\|^2 &= -(u\cdot\nabla w, w) - (w\cdot\nabla u, w) - (w\cdot\nabla w, w)\\ &\leqslant \frac{\nu}{2}\|\nabla w\|^2 + \frac{c}{\nu}\sup|u|^2\|w\|^2 + \frac{c}{\nu^3}\|\nabla u\|^4\|w\|^2 + c_8\|w\|\|\nabla w\|^2,\end{aligned}$$

using the second, two- or three-dimensional version of (66) in estimating the last term. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + (\nu - c_8\|w\|)\|\nabla w\|^2 \leqslant c_9\left(\frac{1}{\nu}\sup|u|^2 + \frac{1}{\nu^3}\|\nabla u\|^4\right)\|w\|^2. \tag{68}$$

Therefore, on any interval during which $\|w\| \leqslant \nu/2c_8$, Poincaré's inequality $\|w\| \leqslant c_{10}\|\nabla w\|$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \left(\frac{\nu}{2c_8c_{10}^2} - \frac{c_9}{\nu}\sup|u|^2 - \frac{c^9}{\nu^3}\|\nabla u\|^4\right)\|w\|^2 \leqslant 0. \tag{69}$$

Uniqueness and continuous dependence on the data can be easily deduced from (68). From (69) we can see that small perturbations $w(t)$ of $u(t)$ decay exponentially if $u(t)$ is small.

All the results of Theorems 5 and 6 have analogues for the prescribed net flux problem. In proving them, we do not estimate u directly, but rather the term denoted by v in problem (15). That is, we fix the choice of smooth solenoidal flux carriers $\mathbf{b}_2$ and $\mathbf{b}_3$ and then rewrite (15b) in terms of the new unknown v. This introduces many additional terms of the forms $F_i(\mathbf{b}_i \cdot \nabla \mathbf{v}, \varphi)$, $F_i(\mathbf{v} \cdot \nabla \mathbf{b}_i, \varphi)$, $F_i(\nabla \mathbf{b}_i, \nabla\varphi)$ and $F_iF_j(\mathbf{b}_i \cdot \nabla \mathbf{b}_j, \varphi)$, but none of these causes essential new difficulties. Even for large data one can estimate them by a well-known technique of Leray and Hopf. However, for large data the term $(\mathbf{v} \cdot \nabla v, \varphi)$ causes the same difficulties as the term $(u \cdot \nabla u, \varphi)$ does for problems with prescribed pressure drops. The Stokes operator for the flux problem is defined just as for the pressure problem, except that one uses the basic space $J_1(\Omega)$ in place of $J_1^*(\Omega)$. In prescribing an initial value u_0 for problem (15) or its equivalents, one must of course ensure that it satisfies the compatibility conditions

$$\int_{S_i} u_0 \cdot n \,\mathrm{d}S = F_i(0). \tag{70}$$

In summary, we obtain the following.

Theorem 7

If $F \equiv \sum_i |F_i|$ is sufficiently small, there exists a steady smooth solution of problem (15), and of its equivalents (16) and (41). For any smooth $F_i(t)$ and any $u_0 \in J_1^*(\Omega)$ satisfying (70), one has results precisely analogous to those stated for the prescribed pressure drop problem in Theorem 6.

Mathematical theory has been very valuable to computational practice for many years, in many ways and in particular for suggesting variational formulations of problems and the Galerkin method of constructing solutions. The results of this section ensure that the problems proposed in Section 3 are well posed and appropriate for numerical computation, at least in the case of small data. In its present state, however, the existence theory for the Navier–Stokes equations is too rudimentary to serve as a sharp knife giving decisive answers in the case of large data. That is especially true when one sets free boundary conditions on inflow boundaries. We have not, however, experienced any difficulties in our computations for such problems.

ACKNOWLEDGEMENTS

This work has been supported by the Deutsche Forshungsgemeinschaft (DFG) and the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

1. P. M. Gresho, 'Some current CFD issues relevant to the incompressible Navier–Stokes equations', *Comput. Methods Appl. Mech. Eng.*, **87**, 201–252 (1991).
2. P. M. Gresho and R. L. Sani, 'Résumé and remarks on the open boundary condition minisymposium', *Int. j. numer. methods fluids*, **18**, 983–1008 (1994).
3. S. Turek 'Tools for simulating nonstationary incompressible flow via discretely divergence-free finite element models,' *Int. j. numer. methods fluids*, **18**, 71–105 (1994).
4. J. G. Heywood 'On uniqueness questions in the theory of viscous flow', *Acta Math.*, **136**, 61–102 (1976).
5. J. G. Heywood 'Auxiliary flux and pressure conditions for Navier–Stokes problems', in R. Rautmann (ed.), *Proc. IUTAM Symp.*, Paderborn, 1979, Lecture Notes in Mathematics, Vol. 771, Springer, Berlin, 1980, pp. 223–234.
6. V. A. Solonnikov 'On the solvability of boundary and initial–boundary value problems for the Navier-Stokes system in domains with noncompact boundaries', *Pacific J. Math.*, **93**, 443–458 (1981).
7. V. N. Maslennikova and M. E. Bogovskii, 'Sobolev spaces of solenoidal vector fields', *Sib. Mat. Zh.*, **22**, 91–118 (1981) (Engl. transl. *Sib. Math. J.*, **22**, 399–420 (1982)).
8. G. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. I, *Linearized Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 38, Springer, Berlin, 1994.
9. G. Galdi *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. II, *Nonlinear Steady Problems*, Springer Tracts in Natural Philosophy, Vol. 39, Springer, Berlin, 1994.
10. V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer, Berlin, 1986.
11. O. Pironneau, *Finite Element Methods for Fluids*, Wiley, New York, 1983.
12. G. Prodi, 'Theoremi di tipo locale per il sistema di Navier–Stokes e stabilitadella soluzioni stazionarie', *Rend. Sem. Mat. Univ. Padova*, **32**, 374–397 (1962).
13. J. G. Heywood 'The Navier–Stokes equations: on the existence, regularity and decay of solutions', *Indiana Univ. Math. J.*, **29**, 639–681 (1980).
14. J. G. Heywood and R. Rannacher, 'Finite element approximation of the nonstationary Navier–Stokes problem. Part 1: Regularity of solutions and second-order error estimates for spatial discretization', *SIAM J. Numer. Anal.*, **19**, 275–311 (1982); 'Part 2: Stability of solutions and error estimates uniform in time', *SIAM J. Numer. Anal.*, **23**, 750–777 (1986); 'Part 3: Smoothing property and higher order error estimates for spatial discretization', *SIAM J. Numer. Anal.*, **25**, 489–512 (1988); 'Part 4: Error analysis for second-order time discretization', *SIAM J. Numer. Anal.*, **27**, 353–384 (1990).
15. J. G. Heywood and W. Xie, Smooth solutions of the vector Burgers equation in nonsmooth domains, preprint 1995.
16. W. Xie, Integral representations and L^∞ bounds for solutions of the Helmholtz equation on arbitrary open sets in $\mathbf{R}^2$ and $\mathbf{R}^3$. *Diff. Int. Eqn*, **8**, 689–698 (1995).